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# Continuum Regularization of Gauge Theory with Fermions

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**Abstract.** The covariant-derivative regularization program is discussed for $d$-dimensional gauge theory coupled to fermions in an arbitrary representation.

## 1. Introduction

A covariant-derivative regularization program for continuum quantum field theory has recently been proposed [1–4]. The scalar prototype [2], scalar electrodynamics [3], and Yang-Mills gauge theory [1, 3–5] have been analyzed in detail. The program is designed to provide adequate and presumably non-perturbative regularization for any theory of interest, including theories of supersymmetry and general coordinate invariance, so many applications remain to be studied. Here we apply the program to gauge theory with fermions.

The extension to include fermions is straightforward. As usual, the regularization may be studied either at the $(d+1)$-dimensional stochastic level, via Markovian-regularized Langevin systems, or at the $d$-dimensional level of the regularized Schwinger-Dyson equations. We provide details at both levels, emphasizing special fermionic features.

The two-noise equations developed by Sakita [6], Ishikawa [7] and Alfaro and Gavela [8] (SIAG equations) have emerged as an adequate fermionic extension of the Parisi-Wu program [9]. These equations provide an almost bosonic stochastic description of fermions, and have been applied, with stochastic regularization by fifth-time smearing [10, 11], to the study of anomalies in background gauge fields [8, 12–14] and vacuum polarization in QED [14]. We adopt these SIAG equations here as an adequate vehicle for our regularization at the stochastic level. It should be emphasized however that other satisfactory stochastic formulations of fermions exist, such as that developed for numerical purposes in [15], and these also may be studied with our regularization. We also continue to employ Zwanziger's gauge-fixing [16, 11, 17] which, in conjunction with our regularization, provides an apparently non-perturbative description of QCD.

The organization of the paper is as follows. We begin in Sect. 2 at the $(d+1)$-dimensional level, giving the regularized Langevin systems for $d$-dimensional gauge theory coupled to fermions in any representation. The weak coupling expansion of these systems is discussed, and finally summarized in Sect. 3 by a set of tree rules for the construction of the regularized Langevin diagrams to all orders.

In Sect. 4, we apply the rules in a computation of the leading fermionic contribution to the gluon vacuum polarization in four dimensions (QCD). The fermionic contribution to the gluon mass is zero, verifying gauge-invariance of the regularized systems. The leading term exhibits the Zwanziger non-transversality (obtained also with dimensional regularization) previously observed in the Yang-Mills contribution [18, 3, 4]. We also verify the smooth approach of the Zwanziger gauge-fixed two-point Green function to the ordinary Landau gauge value [19]. For completeness, the analogous results for scalar electrodynamics [3] are also given.

Section 5 is an exposition of the regularization at the $d$-dimensional level, in terms of the regularized Schwinger-Dyson (SD) equations. Fermionic contributions to renormalization constants are computed in the SD renormalization scheme [2, 4]. Finally, we mention an alternate "naive" regularized SD formulation of gauge theory with fermions.

In the last Sect. 6, we study regularized fermions in background gauge fields. These systems provide adequate modelling for axial and chiral anomalies, but cannot be considered totally satisfactory as models of gauge coupling: In contrast to the regularized dynamical gauge theories, the problem of Lee and Zinn-Justin [20] is not removed in background fields.

## 2. Regularized Langevin Systems for Gauge Theory with Fermions

In this section, we discuss covariant-derivative regularization at the $(d+1)$-dimensional stochastic level for $d$-dimensional $SU(N)$ gauge theory coupled to Dirac fermions[1] in an arbitrary representation $R$ of the gauge group. The Euclidean action is

$$S=\int(dx)[\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\bar\psi^A_i(\not{D}^{AB}_{ij}+\delta^{AB}\delta_{ij}m)\psi^B_j] \tag{2.1}$$

where $(dx)\equiv d^d x$ and $F^a_{\mu\nu}(A)$ is the usual Yang-Mills field strength, as a function of the gauge field $A^a_\mu$. Our fermionic notation is as follows. The Dirac fields $\psi^A_i$, $\bar\psi^B_j$ carry spinor sub-indices and capital letters which run over the representation, while the Dirac matrices $(\gamma_\mu)_{ij}$ and representation matrices $(T^a)^{AB}$ satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\qquad \gamma^\dagger_\mu=\gamma_\mu, \tag{2.2a}$$

$$[-iT^a,-iT^b]=f^{abc}(-iT^c),\qquad (T^a)^\dagger=T^a, \tag{2.2b}$$

$$\mathrm{Tr}[T^a T^b]=C_R\,\delta^{ab}, \tag{2.2c}$$

where $C_R$ is the Dynkin index of representation $R$. Furthermore,

$$D^{ab}_\mu\equiv\delta^{ab}\partial_\mu+g f^{abc}A^c_\mu \tag{2.3a}$$

$$(\not{D})^{AB}_{ij}\equiv(\gamma_\mu)_{ij}\vec{D}^{AB}_\mu,\qquad \vec{D}^{AB}_\mu\equiv\delta^{AB}\vec\partial_\mu+ig\,A^a_\mu(T^a)^{AB} \tag{2.3b}$$

$$(\overleftarrow{\not{D}})^{AB}_{ij}\equiv(\gamma_\mu)_{ij}(\overleftarrow{D}^\dagger)^{AB}_\mu,\qquad (\overleftarrow{D}^\dagger)^{AB}_\mu\equiv\delta^{AB}\overleftarrow\partial_\mu-ig\,A^a_\mu(T^a)^{AB} \tag{2.3c}$$

are the relevant covariant derivatives.

An adequate vehicle for the regularization of such theories at the stochastic level is the set of regularized and Zwanziger gauge-fixed SIAG-Langevin systems,

$$\dot A^a_\mu(x,t)=-\frac{\delta S}{\delta A^a_\mu}(x,t)+D^{ab}_\mu Z^b(x,t)+\int(dy)\,R^{ab}_{xy}\,\eta^b_\mu(y,t), \tag{2.4a}$$

$$\begin{aligned}\dot\psi^A_i(x,t)=&(\not{D}^2_x-m^2)^{AB}_{ij}\psi^B_j(x,t)+\int(dy)(\mathbb{R}^{AB}_{ij})_{xy}(\eta^B_2)_j(y,t)\\&-(\not{D}_x-m)^{AB}_{ij}\int(dy)(\mathbb{R}^{BC}_{jl})_{xy}(\eta^C_1)_l(y,t)\\&-ig\,Z^a(T^a)^{AB}\psi^B_i(x,t),\end{aligned} \tag{2.4b}$$

$$\begin{aligned}\dot{\bar\psi}^A_i(x,t)=&\bar\psi^B_j(x,t)(\overleftarrow{\not{D}}^2_x-m^2)^{BA}_{ji}+\int(dy)(\bar\eta^B_1)_j(y,t)(\mathbb{R}^{BA}_{ji})_{yx}\\&+\int(dy)(\bar\eta^C_2)_l(y,t)(\mathbb{R}^{CB}_{lj})_{yx}(\overleftarrow{\not{D}}_x+m)^{BA}_{ji}\\&+ig\,Z^a(T^a)^{BA}\bar\psi^B_i(x,t),\end{aligned} \tag{2.4c}$$

in which the various Gaussian noise fields satisfy

$$\langle\eta^a_\mu(x,t)\,\eta^b_\nu(x',t')\rangle_\eta=2\delta^{ab}\delta_{\mu\nu}\delta(t-t')\delta(x-x') \tag{2.5a}$$

$$\langle(\eta^A_\alpha)_i(x,t)(\bar\eta^B_\beta)_j(x',t')\rangle_\eta=\delta_{\alpha\beta}\delta^{AB}\delta_{ij}\delta(t-t')\delta(x-x'). \tag{2.5b}$$

Here $\eta_1$, $\eta_2$, $\bar\eta_1$, $\bar\eta_2$ are Grassmann variables which anticommute among one another. For computational purposes, we shall choose the Zwanziger gauge-fixing $Z^a=\alpha^{-1}\partial\cdot A^a$, and we will check below at the Schwinger-Dyson level that gauge-invariant quantities are independent of the gauge-fixing.

The Yang-Mills regulator $R(\Delta)$, a function of the covariant Laplacian $\Delta=D^2$, has been discussed in detail in [1, 3, 5]. Here, we have also introduced a fermionic regulator $[\mathbb{R}^{AB}_{ij}(\not{D}^2)]_{xy}$, which is a function of the covariant fermionic Laplacian $\not{D}^2$. The explicit form of the matrix elements of this Laplacian

$$[(\not{D}^2)^{AB}_{ij}]_{xy}=(\not{D}^2_x)^{AB}_{ij}\,\delta(x-y), \tag{2.6a}$$

$$(\not{D}^2_x)^{AB}_{ij}=\delta^{AB}\delta_{ij}\partial^2_x+g\,\Gamma^{(1)AB}_{ij}(x)+g^2\,\Gamma^{(2)AB}_{ij}(x), \tag{2.6b}$$

$$\Gamma^{(1)AB}_{ij}\equiv i(T^a)^{AB}\{(-i\sigma_{\mu\nu})_{ij}(\partial_\mu A^a_\nu)+\delta_{ij}[(\partial\cdot A^a)+2A^a\cdot\partial]\}, \tag{2.6c}$$

$$\Gamma^{(2)AB}_{ij}\equiv-(T^a T^b)^{AB}\{-i(\sigma_{\mu\nu})_{ij}A^a_\mu A^b_\nu+\delta_{ij}A^a\cdot A^b\}, \tag{2.6d}$$

$$\sigma^\dagger_{\mu\nu}=\sigma_{\mu\nu}\equiv\tfrac{i}{2}[\gamma_\mu,\gamma_\nu] \tag{2.6e}$$

will be useful below. The theory is safely regularized to all orders in any dimension with heat-kernel regularization [5] for both regulators: $R=\exp(\Delta/\Lambda^2)$, $\mathbb{R}=\exp(\not{D}^2/\Lambda^2)$. When explicit Feynman rules are required however for four-dimensional applications (QCD), we will choose minimal power-law regularization

$$R=(1-\Delta/\Lambda^2)^{-2},\qquad \mathbb{R}=(1-\not{D}^2/\Lambda^2)^{-1} \tag{2.7}$$

as in [3].

To study the weak-coupling expansions of the Langevin systems, we consider the set of equivalent integral equations,

$$\begin{aligned}A^a_\mu(x,t)=&\int_{-\infty}^{t}dt'\int(dy)\,G^{ab}_{\mu\nu}(x-y,t-t')[V^b_\nu(y,t')+J^b_\nu(y,t')\\&+\frac{1}{\alpha}Y^b_\nu(y,t')+\int(dz)\,R^{bc}_{yz}\,\eta^c_\nu(z,t')],\end{aligned} \tag{2.8a}$$

$$\begin{aligned}\psi^A_i(x,t)=&\int_{-\infty}^{t}dt'\int(dy)\,G^{AB}_{ij}(x-y,t-t')\\&\times\{[\Gamma^{BC}_{jk}(y,t')-\frac{1}{\alpha}\tilde Y^{BC}_{jk}(y,t')]\psi^C_k(y,t')\\&-(\not{D}_y-m)^{BC}_{jk}\int(dz)(\mathbb{R}^{CD}_{kl})_{yz}(\eta^D_1)_l(z,t')\\&+\int(dz)(\mathbb{R}^{BC}_{jk})_{yz}(\eta^C_2)_k(z,t')\},\end{aligned} \tag{2.8b}$$

[1] The case of Weyl fermions, e.g. the Weinberg-Salam model, is also straightforward. Chiral anomalies are discussed in Sect. 6

$$\bar{\psi}_i^A(x,t) = \int_{-\infty}^{t} dt' \int (dy)\, G_{ji}^{BA}(x-y,t-t')$$
$$\times \{[(\Gamma^\dagger)_{kj}^{CB}(y,t') + \frac{1}{\alpha}\, \tilde{Y}_{kj}^{CB}(y,t')]\, \bar{\psi}_k^C(y,t')$$
$$+ \int (dz)(\bar{\eta}_2^D)_l(z,t')(\mathbb{R}_{lk}^{DC})_{zy}(\bar{\not{D}}_y + m)_{kj}^{CB} \qquad (2.8\,\mathrm{c})$$
$$+ \int (dz)(\bar{\eta}_1^C)_k(z,t')(\mathbb{R}_{kj}^{CB})_{zy}\}.$$

In these equations, we have employed the usual gauge-field Green function

$$G_{\mu\nu}^{ab}(x-y,t-t') = \delta^{ab}\,\theta(t-t') \int (dp)\, \mathrm{e}^{-ip\cdot(x-y)} \times [T_{\mu\nu}\, \mathrm{e}^{-p^2(t-t')} + L_{\mu\nu}\, \mathrm{e}^{-p^2(t-t')/\alpha}], \qquad (2.9)$$

and the (SIAG-bosonized) fermionic Green function

$$G_{ij}^{AB}(x-y,t-t') = \delta^{AB}\,\delta_{ij}\,\theta(t-t') \int (dp)\, \mathrm{e}^{-ip\cdot(x-y)}\, \mathrm{e}^{-(p^2+m^2)(t-t')}, \qquad (2.10)$$

where $(dp) \equiv d^d p/(2\pi)^d$. The interaction terms are defined as follows.

$$V_\nu^b \equiv -g f^{bcd}[\partial_\sigma(A_\sigma^c A_\nu^d) - (\partial_\sigma A_\nu^c) A_\sigma^d + (\partial_\nu A_\sigma^c) A_\sigma^d] - g^2 f^{bcd} f^{cfe} A_\sigma^f A_\nu^e A_\sigma^d, \qquad (2.11\,\mathrm{a})$$

$$Y_\nu^b \equiv g f^{bcd} A_\nu^d (\partial\cdot A^c), \qquad (2.11\,\mathrm{b})$$

$$J_\nu^b \equiv -ig\, \bar{\psi}^A \gamma_\nu (T^b)^{AB} \psi^B, \qquad (2.11\,\mathrm{c})$$

$$\Gamma_{ij}^{AB} \equiv [g\Gamma^{(1)} + g^2\Gamma^{(2)}]_{ij}^{AB}, \qquad (2.11\,\mathrm{d})$$

$$\tilde{Y}_{ij}^{AB} \equiv ig\, \delta_{ij} (\partial\cdot A^a)(T^a)^{AB}, \qquad (2.11\,\mathrm{e})$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are defined in Eq. (2.6), and $(\Gamma^\dagger)_{kj}^{CB} \equiv (\Gamma_{jk}^{BC})^*$.

As usual, we also need to expand the regulators into regulator strings. Such expansions have been discussed in detail in [1, 3, 5] for the Yang-Mills regulator. In the same way, corresponding expansions of the fermionic regulators are obtained,

$$\mathbb{R}(\not{D}^2) = \left[1 - \frac{\not{D}^2}{\Lambda^2}\right]^{-1} = \sum_{n=0}^{\infty} \left[\frac{1}{1-\Box/\Lambda^2}\left(\frac{g\Gamma^{(1)} + g^2\Gamma^{(2)}}{\Lambda^2}\right)\right]^n \frac{1}{1-\Box/\Lambda^2}. \qquad (2.12)$$

The objects $\Gamma^{(1)}$ and $\Gamma^{(2)}$ therefore serve a dual role, entering first in the SIAG structure of Eq. (2.8), and now also as the one- and two-gluon regulator vertices, which connect regulator strings.

The integral equations (2.8) may then be more or less conventionally expanded to all orders as Langevin tree graphs. In the next section, we give the Langevin-Feynman rules for the construction of these tree graphs to all orders.

## 3. Regularized Langevin Tree Rules and Diagrams

The pure Yang-Mills part of these rules has been discussed in [1, 3]. Here we study only the fermionic additions. In Fig. 1, we have given the new rules explicitly for the regulators of Eq. (2.7), which are minimal in $d=4$. Throughout these rules, thick arrows (——▶——) indicate the direction of decreasing fifth-time, while thin arrows (——>——) track the direction of fermionic charge flow.

### *Propagators*

The two new fermionic propagators of the theory are shown in Fig. 1a: The thin lines are fermionic Green functions $G_{ij}^{AB}$, and the thick lines are fermionic regulator propagators.

### *Noise Vertices*

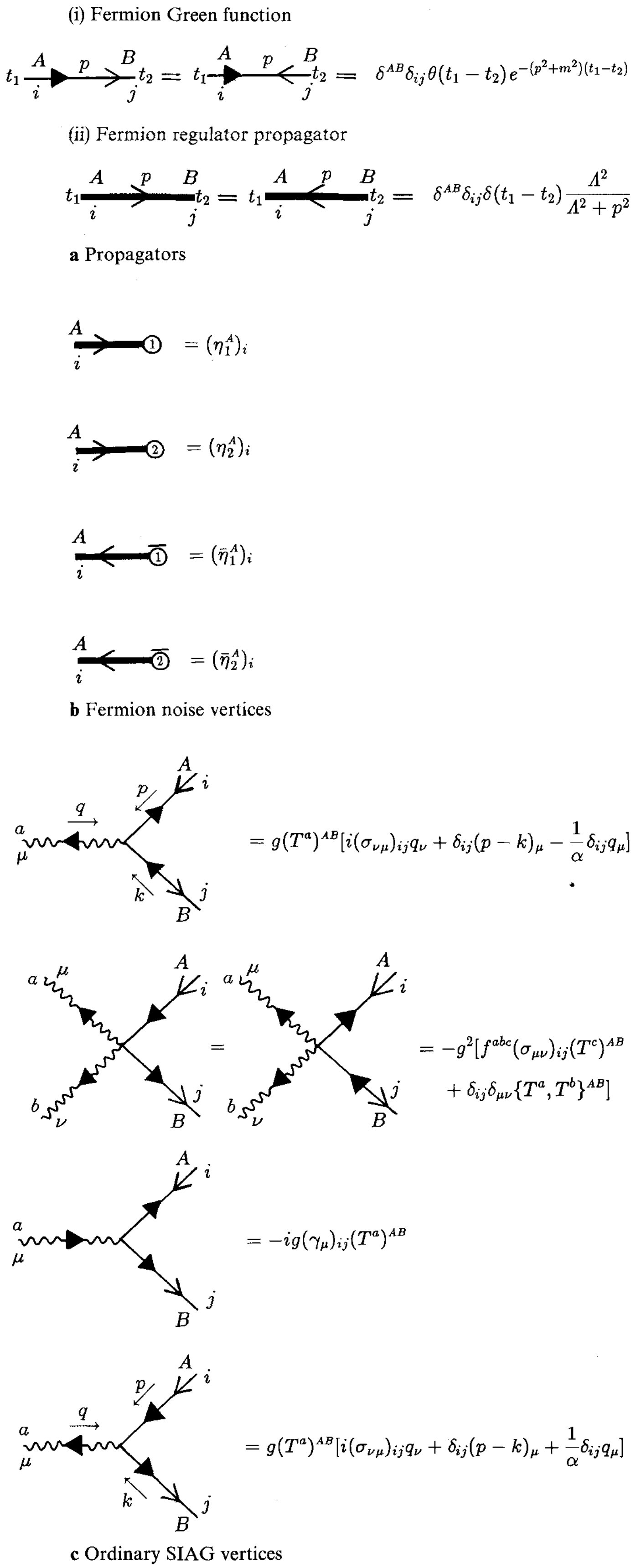
The one-point noise vertices ①, ②, $\bar{①}$, $\bar{②}$, shown in Fig. 1b, are quadrupled relative to familiar cases, since the Grassmann noise is complex and comes in two varieties.

### *Ordinary SIAG Vertices*

Ordinary three- and four-point vertices (with no regulator contributions) are shown in Fig. 1c: The first three-point vertex carries no Zwanziger gauge-fixing, since it arises from the fermionic part of the $\delta S/\delta A$ term in the Langevin equation for $A_\mu^a$. The rest of the ordinary vertices are peculiar to the SIAG form, arising from the $\not{D}^2\psi$, $\bar{\psi}\bar{\not{D}}^2$ terms, plus the Zwanziger term in the fermion equations.

### *Joining Vertices*

These vertices, shown in Fig. 1d, join the regulator strings to the rest of a diagram. The first two-point joining vertex, which comes from regulator factors times $\bar{\eta}_1$ and $\eta_2$, occurs regularly in previous work [1–3], but there is an extra two-point joining vertex (∅) which comes from the $(\not{D}-m)\,\mathbb{R}\eta_1$, $\bar{\eta}_2\,\mathbb{R}(\bar{\not{D}}+m)$ terms. The joining vertex ∅ is always connected by a regulator propagator to ① or $\bar{②}$, and never to ② or $\bar{①}$. Finally, because the extra SIAG derivatives on $\eta_1$ and $\bar{\eta}_2$ are covariant, there is also a three-point joining vertex.



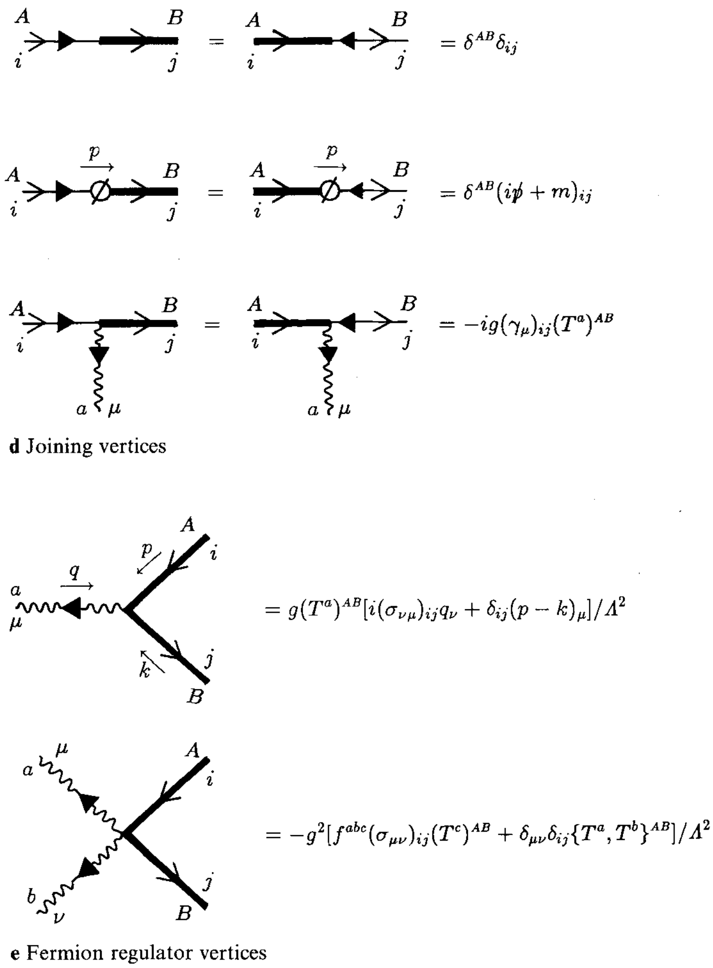




**Fig. 1a–e.** Fermion additions to regularized Langevin tree rules

## *Regulator Vertices*

These two vertices, shown in Fig. 1e, arise in familiar fashion from the regulator expansion, and correspond to the factors $\Gamma^{(1)}$, $\Gamma^{(2)}$ of Eqs. (2.6) and (2.12). As in the case of Yang-Mills [1, 3], the regulator vertices reflect the non-abelian structure of the regulator, and always play a crucial role in maintaining gauge-invariance. On the other hand, as discussed in [3, 4], the explicit $\Lambda^{-2}$ factor of the regulator vertices means that they play essentially no role in the study of one-loop renormalization.

To form the Langevin diagrams, we contract the Langevin trees, using Eq. (2.5) as usual. Of course ① [②] noise vertices must only contract to ① [②] noise vertices, as shown in Fig. 2. We shall place a symbol [1] [[2]] at each such contraction. The Grassmann character of the fermionic noise leads naturally to a factor of (−1) for each closed fermion loop and a global sign [21] in the Langevin diagrams. An over-

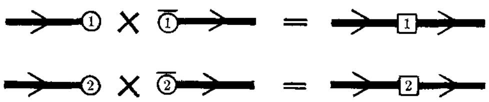
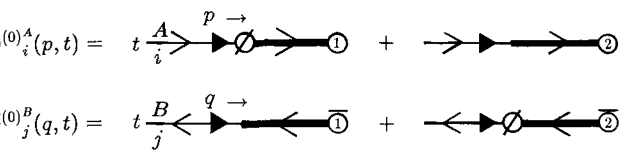
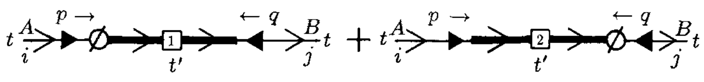


**Fig. 2.** Contractions of fermionic noise



**a** Zeroth-order fermion field tree graphs



**b** Diagrams for $\langle \psi_i^A(p)\, \bar{\psi}_j^B(q)\rangle^{(0)}$

**Fig. 3a, b.** Fermionic two-point function

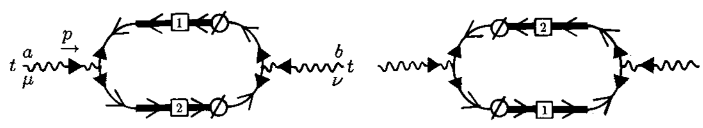


**Fig. 4.** Diagrams with vanishing contributions to the gluon mass

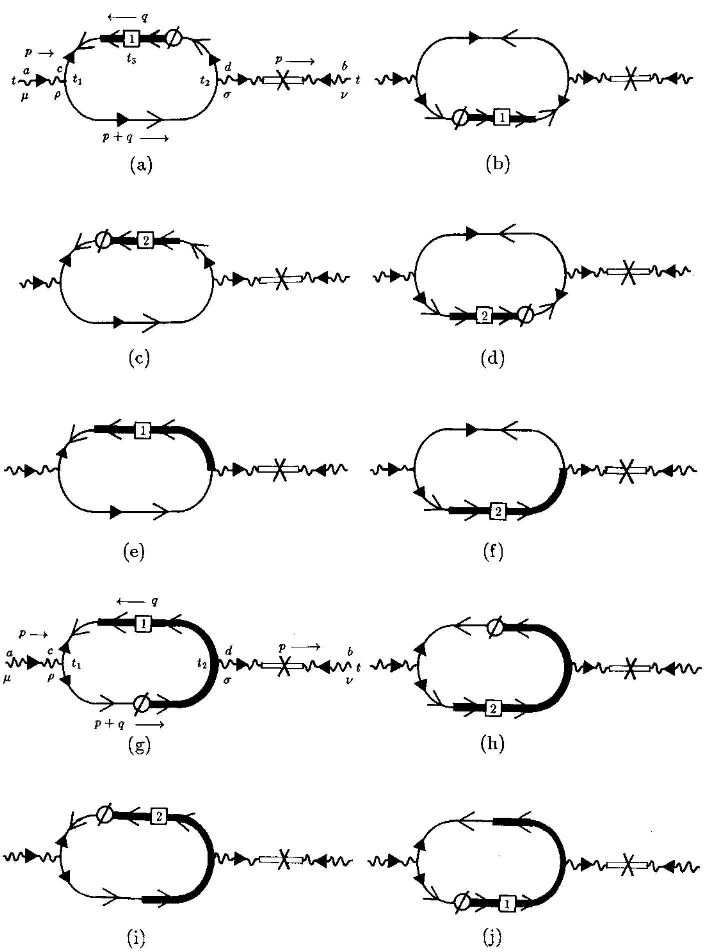


**Fig. 5.** Diagrams with non-vanishing individual contributions to the gluon mass. Indices and momenta are given for the examples in the text

all momentum-conservation factor $(2\pi)^d\, \delta\left(\sum_{i=1}^{n} p_i\right)$ should also be included in Langevin diagrams for $n$-point functions.

As the simplest illustration of these rules, we compute the zeroth-order two-point fermion Green functions. Figure 3a gives the tree-diagrammatic representation of the zeroth order fields $\psi$ and $\bar{\psi}$. Contracting as shown in Fig. 3b, we obtain for the regularized $\psi\bar{\psi}$ propagator

$$\begin{aligned}
&\langle \psi_i^A(p)\, \bar{\psi}_j^B(q)\rangle^{(0)} \\
&= 2\delta^{AB}(2\pi)^d\, \delta(p+q)\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^2 \int_{-\infty}^{t} dt'\, e^{-2(p^2+m^2)(t-t')}(i\not{p}+m)_{ij} \\
&= \delta^{AB}(2\pi)^d\, \delta(p+q)\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^2 \left(\frac{1}{-i\not{p}+m}\right)_{ij},
\end{aligned} \quad (3.1)$$

and same result with a minus sign for $\langle \bar{\psi}_j^B(q)\, \psi_i^A(p)\rangle^{(0)}$, according to the global sign rule above.

## 4. Fermionic Contribution to the QCD Vacuum Polarization

As a non-trivial check on the gauge-invariance of the regularized Langevin systems above, we use the diagrammatic rules of the previous section to compute the fermionic contribution to the gluon vacuum polarization in four dimensions. In particular, since the regularized Yang-Mills contribution to the gluon mass is zero [1, 3], the fermionic contribution must also vanish.

There are altogether 24 diagrams with one internal fermion loop which contribute to the gauge-field propagator $\langle A_\mu^a(x)\, A_\nu^b(y)\rangle$ and hence to the vacuum polarization $\Pi_{\mu\nu}^{ab}(p)$. The first 14 of these are shown in Figs. 4 and 5, while the remaining 10 may be trivially obtained from those of Fig. 5 by interchanging $(a, \mu, p)$ with $(b, \nu, -p)$. Before beginning the computations, it is instructive to discuss some qualitative features of these diagrams.

*SIAG Structure*

The four diagrams of Fig. 4 and the first $6\times 2=12((a)-(f))$ diagrams of Fig. 5 comprise 16 ordinary diagrams, attributable to the SIAG structure, which contain no regulator vertices. In the naive regulator limit, $(R=\mathbb{R}=1)$, we have checked explicitly that these 16 diagrams combine to form the single usual vacuum polarization Feynman diagram. Our regulator is responsible for the additional $4\times 2=8((g)-(j))$ diagrams of Fig. 5, which contain explicit regulator vertices.

*Mass and $p^2 \ln \Lambda^2$ Contributions*

As discussed in [3, 4], the 4 diagrams of Fig. 4 fail to contribute to the gluon mass on dimensional grounds, since they contain no contractions on external lines. On the other hand, as noted above, the explicit $\Lambda^{-2}$ of the regulator vertices means that the regulator vertex diagrams $(g)-(j)$ of Fig. 5 make no contributions to wavefunction and $\alpha$ renormalizations.

In the computations below, truncation to define $\Pi^{ab}_{\mu\nu}(p)$ at large $\Lambda$ are accomplished by factoring out two zeroth-order (Zwanziger gauge-fixed) propagators

$$\frac{1}{p^2}[T_{\mu\rho}(p)+\alpha L_{\mu\rho}(p)]\times\frac{1}{p^2}[T_{\nu\sigma}(p)+\alpha L_{\nu\sigma}(p)] \tag{4.1}$$

from the propagator diagrams for $\langle A^a_\rho A^b_\sigma\rangle$.

As an explicit example of an ordinary diagram, we use the Langevin rules of Fig. 1 and [1, 3] to write down the expression for diagram 5(a),

$$\begin{aligned}&-g^2 C_R \delta^{AB} f\left[\frac{1}{p^2}\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^4\right]\\&\int(dq)\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt_3\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^2\\&\times\left(\mathrm{Tr}[\gamma_\rho \not p\gamma_\sigma \not q]+4q_\rho\left(2q-\frac{1}{\alpha}p\right)_\sigma\right)\\&\times e^{-(q^2+m^2)(t_1-t_3)}\,e^{-(q^2+m^2)(t_2-t_3)}\,e^{-[(p+q)^2+m^2](t_1-t_2)}\\&\times\{T_{\mu\rho}(p)\,T_{\nu\sigma}(p)\,e^{-p^2(t-t_1)}\,e^{-p^2(t-t_2)}\\&+\alpha L_{\mu\rho}(p)\,L_{\nu\sigma}(p)\,e^{-p^2(t-t_1)/\alpha}\,e^{-p^2(t-t_2)/\alpha}\},\end{aligned} \tag{4.2}$$

where $f$ is the number of flavors. After fifth-time integration, this becomes

$$\begin{aligned}&-\frac{g^2}{4}C_R\delta^{ab}f\left[\frac{1}{p^4}\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^4\right]\\&\int(dq)\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^2\left(\mathrm{Tr}[\gamma_\rho \not p\gamma_\sigma \not q]+4q_\rho\left(2q-\frac{1}{\alpha}p\right)_\sigma\right)\\&\times\frac{1}{q^2+m^2}\left\{\frac{T_{\mu\rho}(p)\,T_{\nu\sigma}(p)}{Q(1)}+\frac{\alpha^2 L_{\mu\rho}(p)\,L_{\nu\sigma}(p)}{Q(\alpha)}\right\},\end{aligned} \tag{4.3}$$

in which we have defined the quantity

$$Q(\alpha)\equiv(q^2+m^2)+[(p+q)^2+m^2]+p^2/\alpha. \tag{4.4}$$

Integrating also over the internal momentum $q$ gives, after truncation, the following contribution to the zero momentum vacuum polarization

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(0)|_{5(a)}&=\frac{1}{4}\delta_{\mu\nu}\mathcal{N}^{ab}\left(-\Lambda^2+2m^2\ln\frac{\Lambda^2}{m^2}-3m^2\right)\\&\quad+\text{terms which vanish as }\Lambda\to\infty.\end{aligned} \tag{4.5}$$

Here we have defined the constant

$$\mathcal{N}^{ab}\equiv g^2 C_R\,\delta^{ab} f/(4\pi)^2 \tag{4.6}$$

and the neglected terms are of order $m^4/\Lambda^2$ times possible logarithms. The $p^2 \ln \Lambda^2$ contribution to $\Pi^{ab}_{\mu\nu}(p)$ may also be computed by differentiation with respect to external momentum[2]. These results are recorded in Table 1.

As an example of a diagram with a regulator vertex, we also give the explicit expression for diagram 5(g).

$$\begin{aligned}&\frac{ig^2 C_R\delta^{ab} f}{\Lambda^2}\left[\frac{1}{p^2}\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^4\right]\\&\times\int(dq)\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^2\left(\frac{\Lambda^2}{\Lambda^2+(p+q)^2}\right)\\&\times\mathrm{Tr}[\gamma_\rho(i\not p+i\not q+m)\{(2q+p)_\sigma+i\sigma_{\gamma\sigma}p_\gamma\}]\\&\times e^{-(q^2+m^2)(t_1-t_2)}\,e^{-[(p+q)^2+m^2](t_1-t_2)}\\&\times\{T_{\mu\rho}(p)\,T_{\nu\sigma}(p)\,e^{-p^2(t-t_1)}\,e^{-p^2(t-t_2)}\\&+\alpha L_{\mu\rho}(p)\,L_{\nu\sigma}(p)\,e^{-p^2(t-t_1)/\alpha}\,e^{-p^2(t-t_2)/\alpha}\}\end{aligned} \tag{4.7a}$$

$$\begin{aligned}&=-\frac{g^2 C_R\delta^{ab} f}{2\Lambda^2}\left[\frac{1}{p^2}\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^4\right]\\&\int(dq)\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^2\left(\frac{\Lambda^2}{\Lambda^2+(p+q)^2}\right)\\&\times\mathrm{Tr}[\gamma_\rho(\not p+\not q)\{(2q+p)_\sigma+i\sigma_{\gamma\sigma}p_\gamma\}]\\&\times\left\{\frac{T_{\mu\rho}(p)\,T_{\nu\sigma}(p)}{Q(1)}+\frac{\alpha^2 L_{\mu\rho}(p)\,L_{\nu\sigma}(p)}{Q(\alpha)}\right\}.\end{aligned} \tag{4.7b}$$

[2] In fact, there is a non-uniformity in the computation of the $p^2 \ln\Lambda^2$ terms of $\Pi^{ab}_{\mu\nu}$ at $\alpha=0$, since closer examination reveals dependence on parameters such as $\ln(\alpha\Lambda^2/p^2)$. The results quoted for these terms are, strictly speaking, valid only for $\alpha\neq 0$

**Table 1.** Order $g^2$ contributions to leading terms of $\Pi^{ab}_{\mu\nu}(p)$ in units of $\mathcal{N}^{ab} \equiv g^2 C_R \delta^{ab} f/(4\pi)^2$. Diagrams with identical contributions are grouped together in a row and their sums are listed.

| Diagrams | Contributions to $\Pi^{ab}_{\mu\nu}(p)$ |
|---|---|
| 4(a), 4(b), 4(c), 4(d) | $p^2[T_{\mu\nu}+(1/\alpha) L_{\mu\nu}] \ln(\Lambda^2/m^2)$ |
| 5(a), 5(b), 5(c), 5(d) plus $(\mu, a, p) \leftrightarrow (\nu, b, -p)$ | $\delta_{\mu\nu}[-2\Lambda^2+4m^2 \ln(\Lambda^2/m^2)-6m^2]$ $+p^2[(2/3) T_{\mu\nu}+L_{\mu\nu}] \ln(\Lambda^2/m^2)$ |
| 5(e), 5(f) plus $(\mu, a, p) \leftrightarrow (\nu, b, -p)$ | $\delta_{\mu\nu}[4\Lambda^2-4m^2 \ln(\Lambda^2/m^2)+4m^2]$ $+p^2[-3T_{\mu\nu}-(1+(2/\alpha)) L_{\mu\nu}]$ $\ln(\Lambda^2/m^2)$ |
| 5(g), 5(h), 5(i), 5(j) plus $(\mu, a, p) \leftrightarrow (\nu, b, -p)$ | $\delta_{\mu\nu}[-2\Lambda^2+2m^2]$ |

With truncation, we compute

$$\Pi^{ab}_{\mu\nu}(0)|_{5(g)} = \tfrac{1}{4}[-\Lambda^2+m^2] \mathcal{N}^{ab} \delta_{\mu\nu} + \text{terms which vanish as } \Lambda \to \infty. \tag{4.8}$$

As recorded in Table 1, this diagram, (since it contains a regulator vertex), contributes no $p^2 \ln \Lambda^2$ term.

In this way, we have computed all non-zero contributions to $\Pi^{ab}_{\mu\nu}(0)$, and all $p^2 \ln \Lambda^2$ contributions to $\Pi^{ab}_{\mu\nu}(p)$. The results are listed with their diagrams in Table 1. The reader may easily verify that the sum of all contributions to $\Pi^{ab}_{\mu\nu}(0)$ is zero, so the gluon remains massless to this order.

Adding all contributions in Table 1, we obtain the total fermionic contribution to the gluon vacuum polarization

$$\Pi^{(f)ab}_{\mu\nu}(p) = \mathcal{N}^{ab} p^2 \left(-\frac{4}{3} T_{\mu\nu}(p) - \frac{1}{\alpha} L_{\mu\nu}(p)\right) \ln\frac{\Lambda^2}{m^2} + \text{terms finite as } \Lambda \to \infty. \tag{4.9}$$

The transverse term in Eq. (4.9) is the standard [22] fermionic contribution, while the longitudinal term is peculiar to Zwanziger's gauge-fixing, since the same result is obtained in dimensional regularization of the Zwanziger gauge-fixed theory with the dictionary $\ln \Lambda \leftrightarrow (4-d)^{-1}$. This phenomenon was first observed in similar investigations [18, 3] of Yang-Mills theory.

The $\alpha^{-1}$ dependence of the longitudinal term is in a sense an artifact of truncation, however, since the one-loop contribution to the two-point function,

$$\begin{aligned}
&\langle A^a_\mu(p) A^b_\nu(q)\rangle^{(f)} \\
&= (2\pi)^4 \delta(p+q) \left[\frac{T_{\mu\rho}(p)+\alpha L_{\mu\rho}(p)}{p^2}\right] \Pi^{(f)ab}_{\rho\sigma}(p) \\
&\quad \times \left[\frac{T_{\sigma\nu}(p)+\alpha L_{\sigma\nu}(p)}{p^2}\right] \\
&\quad + \text{terms which vanish as } \Lambda \to \infty \\
&= (2\pi)^4 \delta(p+q) \frac{\mathcal{N}^{ab}}{p^2} \left(-\frac{4}{3} T_{\mu\nu} - \alpha L_{\mu\nu}\right) \ln\frac{\Lambda^2}{m^2} \\
&\quad + \text{terms finite as } \Lambda \to \infty
\end{aligned} \tag{4.10}$$

shows a smooth[3] approach to the ordinary Landau gauge result as $\alpha \to 0$. This verifies Zwanziger's formal argument [19] that the gauge-fixing should give standard Landau gauge results as $\alpha \to 0$. Moreover, as we shall see in the next section, the usual $\alpha$-independent fermionic contribution to wavefunction renormalization is obtained.

Although these Zwanziger phenomena have nothing to do with our regularization scheme, we have also computed for completeness the leading term in the case of scalar electrodynamics, using the Langevin rules of [3]. The one-loop contribution is

$$\begin{aligned}
\langle A_\mu(p) A_\nu(p)\rangle &= (2\pi)^4 \delta(p+q) \left[\frac{T_{\mu\rho}(p)+\alpha L_{\mu\rho}(p)}{p^2}\right] \\
&\quad \times \frac{e^2}{(4\pi)^2} p^2 \left[-\frac{1}{3} T_{\rho\sigma} - \frac{1}{2\alpha} L_{\rho\sigma}\right] \ln\frac{\Lambda^2}{m^2} \\
&\quad \times \left[\frac{T_{\sigma\nu}(p)+\alpha L_{\sigma\nu}(p)}{p^2}\right],
\end{aligned} \tag{4.11}$$

which shows the same qualitative features discussed above for fermions. The same behavior is also expected in pure Yang-Mills, when the computations are extended beyond $\alpha = 1$ [18, 3].

## 5. Regularized Schwinger-Dyson Equations and Diagrams

Following standard methods [1–3], but allowing for the Grassmann character of the fermionic noise, the regularized Langevin systems (2.4) may be recast into a set of regularized Schwinger-Dyson (SD) equations[4]

$$\begin{aligned}
0 = \int (dx) \Bigg\langle \Bigg\{ &\Bigg[ -\frac{\delta S}{\delta A^a_\mu(x)} \\
&+ \int (dy)(dz) R^{bc}_{yz} \frac{\delta}{\delta A^c_\mu(z)} R^{ba}_{yx} \Bigg] \frac{\delta}{\delta A^a_\mu(x)} \\
&- \Bigg[ (\not{D}_x - m)^{AB}_{ij} \Bigg( -\frac{\delta S}{\delta \psi^B_j(x)} \\
&+ \int (dy) (\mathbb{R}^2_{xy})^{BC}_{jk} \frac{\delta}{\bar\psi^C_k(y)} \Bigg) \Bigg] \frac{\delta}{\delta \psi^A_i(x)} \\
&- \Bigg[ \Bigg( -\frac{\delta S}{\delta \bar\psi^B_j(x)} + \int (dy) (\mathbb{R}^2_{yx})^{CB}_{kj} \frac{\delta}{\delta \psi^C_k(y)} \Bigg) \\
&\times (\not{D}_x + m)^{BA}_{ji} \Bigg] \frac{\delta}{\delta \bar\psi^A_i(x)} - Z^a(x) G^a(x) \Bigg\} F \Bigg\rangle,
\end{aligned} \tag{5.1}$$

[3] The non-uniformity in $\Pi^{ab}_{\mu\nu}$ at $\alpha = 0$, mentioned in footnote 2, is presumably washed out in the two-point function itself

[4] As discussed in [3, 5], the simpler $\gamma = 0$ form of the regularized Yang-Mills functional Laplacian may also be employed. In this case, the minimal regulators are uniformly $R = (1 - \Delta/\Lambda^2)^{-1}$, $\mathbb{R} = (1 - \not{D}^2/\Lambda^2)^{-1}$ in four dimensions

where

$$G^a(x) \equiv D_\mu^{ab} \frac{\delta}{\delta A_\mu^b(x)} + ig(T^a)^{AB} \psi_i^B(x) \frac{\delta}{\psi_i^A(x)} - ig(T^a)^{BA} \bar{\psi}_i^B(x) \frac{\delta}{\delta \bar{\psi}_i^A(x)} \tag{5.2}$$

is the generator of non-abelian gauge transformations. These SD equations provide a $d$-dimensional description of the regularization scheme which is equivalent[5] to equilibrium results of the regularized Langevin systems, after all fifth-time integrations are performed.

Aside from the Zwanziger term, the SD equations (5.1) are a regularized form of the unregularized second-order SD equations obtainable in the action form as

$$0 = \int \mathscr{D} A \mathscr{D} \bar{\psi} \mathscr{D} \psi \int (dx) \frac{\delta}{\delta A_\mu^a(x)} \left[ \mathrm{e}^{-s} \frac{\delta}{\delta A_\mu^a(x)} F \right], \tag{5.3a}$$

$$0 = -\int \mathscr{D} A \mathscr{D} \bar{\psi} \mathscr{D} \psi \times \int (dx) \left[ (\bar{\not{D}}_x - m)_{ij}^{AB} \frac{\delta}{\delta \bar{\psi}_j^B(x)} \right] \left[ \mathrm{e}^{-s} \frac{\delta}{\delta \psi_i^A(x)} F \right], \tag{5.3b}$$

$$0 = -\int \mathscr{D} A \mathscr{D} \bar{\psi} \mathscr{D} \psi \int (dx) \left[ \frac{\delta}{\delta \psi_j^B(x)} (\bar{\not{D}}_x + m)_{ji}^{BA} \right] \times \left[ \mathrm{e}^{-s} \frac{\delta}{\delta \bar{\psi}_i^A(x)} F \right]. \tag{5.3c}$$

The SD prescription for regularization is then to add equations (5.3a, b, c) and to regularize each of the three functional Laplacians.

The gauge-invariance of the regularized SD systems (5.1) may be verified as in [3]: The crucial ingredients of the demonstration are that 1) The Zwanziger gauge-fixing term vanishes on a gauge-invariant quantity $F_{\mathrm{GI}}$, and 2) The other terms in the SD equations are manifestly gauge-invariant.

As discussed in [1, 2], the SD diagrammatic method is more efficient than Langevin techniques for diagrams with a large number of fifth-time integrations. The rules for the construction of the SD diagrams may be derived directly from the regularized SD equations (5.1) in the manner of the appendix of [2], and appendix C of [3] (or by fifth-time integration of all Langevin diagrams at equilibrium).

As usual, SD diagrams are drawn on the Langevin diagrams. The vertices in the SD formalism are those of the Langevin formalism (Fig. 1), but the Langevin Green functions and fifth-time integrations are replaced by SD "pictures", which provide the momentum denominators called "solid line factors".

[5] As in [2, 3], a unique weak-coupling SD solution is obtained with the boundary condition that the averages have the usual permutation symmetry (anti-symmetry) among the bosonic (fermionic) fields

Since the Yang-Mills SD rules have been thoroughly discussed [3], we concentrate here on the fermionic additions to the rules.

*1)* The fermionic "simple contraction" [3] factor $\delta^{AB} \delta_{ij} \mathbb{R}_0^2(p)/[2(p^2+m^2)]$ includes an additional factor of 1/2.

*2)* No factor of 2 is associated to a fermionic regulator vertex cluster with two incoming lines ($\mathrm{RVC}_2$).

These first two rules follow from the absence of the usual factor of 2 in the charged Grassmann contractions (2.5b). Note also the absence of fermionic regulator vertex clusters with one incoming line ($\mathrm{RVC}_1$'s), since the fermion functional derivatives commute with the fermionic regulator.

*3)* Each fermionic loop gives an additional factor $(-1)$, and the global sign convention holds.

The solid line factors are obtained as usual by studying the operator

$$\mathrm{K} \equiv \int (dp) \left\{ \frac{\delta S_0}{\delta A_\mu^a(-p)} \frac{\delta}{\delta A_\mu^a(p)} + \frac{1}{\alpha} p^2 L_{\mu\nu} A_\nu^a(p) \frac{\delta}{\delta A_\mu^a(p)} + \left[ (i\not{p} + m)_{ij}^{AB} \frac{\delta S_0}{\delta \bar{\psi}_j^B(-p)} \right] \frac{\delta}{\delta \psi_i^A(p)} + \left[ \frac{\delta S_0}{\delta \psi_j^B(-p)} (i\not{p} - m)_{ji}^{BA} \right] \frac{\delta}{\delta \bar{\psi}_i^A(p)} \right\} \tag{5.4}$$

where $S_0$ is the free part of the action. Following Ref. [3], the eigenvectors of K for arbitrary $\alpha$ are

$$F = \bar{\psi}_{i_1}^{A_1}(p_1) \dots \bar{\psi}_{i_m}^{A_m}(p_m) \psi_{j_1}^{B_1}(k_1) \dots \psi_{j_n}^{B_n}(k_n) \times (A^T)_{\mu_1}^{a_1}(q_1) \dots (A^T)_{\mu_r}^{a_r}(q_r) (A^L)_{\nu_1}^{b_1}(r_1) \dots (A^L)_{\nu_s}^{b_s}(r_s), \tag{5.5}$$

where $A^T$ and $A^L$ are the transverse and longitudinal gauge fields. The solid line factors are the inverse eigenvalues of K

$$\mathrm{K} F = \left( \sum_{i=1}^{m} (p_i^2 + m^2) + \sum_{i=1}^{n} (k_i^2 + m^2) + \sum_{i=1}^{r} q_i^2 + \frac{1}{\alpha} \sum_{i=1}^{s} r_i^2 \right) F. \tag{5.6}$$

Note that the SIAG structure gives fermionic contributions to the solid line factors which are totally bosonic, and that the longitudinal gluon terms carry an extra factor $\alpha^{-1}$.

As an explicit example, we demonstrate the SD evaluation of the regulator vertex diagram 5(*g*), which was discussed in Sect. 4. Because $\alpha \neq 1$, there are two SD diagrams, corresponding to both external gluons transverse or both longitudinal. The transverse-longitudinal cross-terms vanish identically. The case with both external gluons longitudinal is shown in Fig. 6 with all relevant indices. The only allowed SD ordering is AB, and the dotted box surrounds the regulator vertex cluster ($\mathrm{RVC}_2$), as discussed in [3].

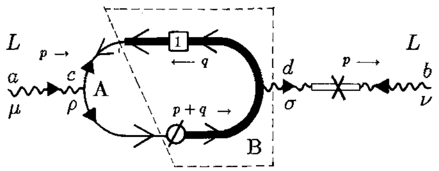



**Fig. 6.** SD diagram for Fig. 5(g) with longitudinal external gluons. The only valid ordering is AB

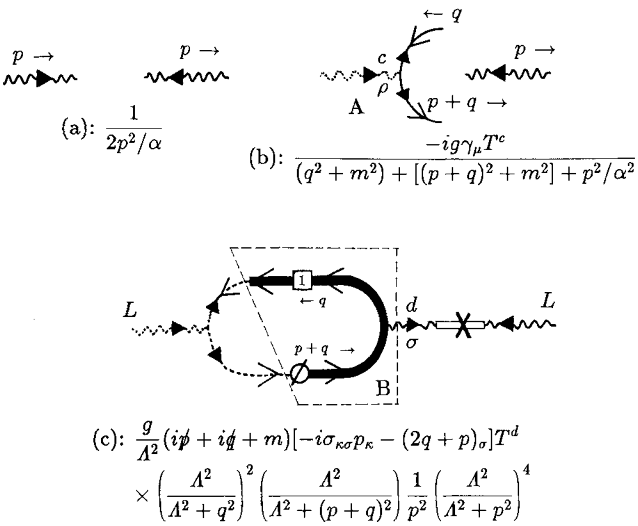



**Fig. 7.** SD pictures for Fig. 6

The sequence of pictures for this diagram is shown in Fig. 7, along with the factor associated with each picture. Collecting all factors from the pictures, a combinatoric factor of unity, a minus sign for the fermion loop, Kronecker-deltas in color indices and longitudinal projection operators for gluon lines, appropriate traces for the fermion loop and a sum over flavors, we find

$$-\frac{g^2 C_R \delta^{ab} f}{2\Lambda^2}\left[\frac{1}{p^4}\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^4\right] \int(dq)\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^2\left(\frac{\Lambda^2}{\Lambda^2+(p+q)^2}\right) \times\frac{\mathrm{Tr}[\gamma_\rho(\not p+\not q)\{(2q+p)_\sigma+i\sigma_{\kappa\sigma}p_\kappa\}]}{q^2+(p+q)^2+2m^2+p^2/\alpha} \times\alpha^2 L_{\mu\rho}(p)\,L_{\nu\sigma}(p) \tag{5.7}$$

as the value of the SD diagram in Fig. 6. This result is precisely the second term in the result Eq. (4.7b), obtained from the Langevin system by fifth-time integration. Exactly the same steps are followed to evaluate the SD diagrams with both external gluons transverse. The only changes are $L\to T$ and $\alpha=1$ in the solid line factors, which gives the first term in Eq. (4.7b).

As an application of the Schwinger-Dyson equations (5.1), we give a brief discussion of renormalization. The SD renormalization program of [4] for pure Yang-Mills is easily extended to include fermions. As the simplest example, the counterterm (CT) contribution to the two-point gluon Green function is easily read off from Fig. 1 of that reference,

$$\mathrm{CT}=\frac{\delta^{ab}}{p^2}[T_{\mu\rho}+\alpha L_{\mu\rho}(p)] \times[(1-Z_A)\,T_{\rho\sigma}(p)+\frac{1}{\alpha}(1-Z_A Z_\alpha)\,L_{\rho\sigma}(p)] \times[T_{\nu\sigma}+\alpha L_{\nu\sigma}(p)]+O\left(\frac{\ln\Lambda}{\Lambda^2}\right). \tag{5.8}$$

Requiring that the renormalized ($R$) contribution (Eq. (4.9)) plus the counterterm contribution (Eq. (5.8)) equals zero gives immediately the fermionic contributions to the renormalization constants

$$(Z_A-1)^{(f)}=-\frac{4}{3}\frac{g^2 C_R f}{16\pi^2}\ln(\Lambda^2/m^2) \tag{5.9a}$$

$$(Z_\alpha-1)^{(f)}=\frac{1}{3}\frac{g^2 C_R f}{16\pi^2}\ln(\Lambda^2/m^2). \tag{5.9b}$$

This is the usual $\alpha$-independent contribution to the wavefunction renormalization, and the fermionic contribution to the $\beta$-function for $\alpha$

$$\beta_\alpha^{(f)}=-\alpha\frac{2}{3}\frac{g^2 C_R f}{16\pi^2} \tag{5.10}$$

exhibits the expected fixed-point at $\alpha=0$.

We have no doubt that the usual fermionic contribution to the coupling constant $\beta$-function is also obtained.

Finally, we mention alternative regularized SD formulations for gauge theory with fermions. As mentioned in [2], SD formulations may exist even when stochastic formulations are questionable. As an example, we consider the "naive" regularized SD equations

$$0=\int(dx)\left\langle\left\{\left[-\frac{\delta S}{\delta A_\mu^a(x)} +\int(dy)(dz)\,R_{yz}^{bc}\frac{\delta}{\delta A_\mu^c(z)}R_{yx}^{ba}\right]\frac{\delta}{\delta A_\mu^a(x)} +\lambda\left[\frac{\delta S}{\delta\bar\psi_i^A(x)}-\int(dy)(\mathbb{R}_{xy}^2)_{ij}^{AB}\frac{\delta}{\bar\psi_j^B(y)}\right]\frac{\delta}{\delta\psi_i^A(x)} -\lambda\left[\frac{\delta S}{\delta\psi_i^A(x)}-\int(dy)(\mathbb{R}_{yx}^2)_{ji}^{BA}\frac{\delta}{\delta\psi_j^B(y)}\right]\frac{\delta}{\delta\bar\psi_i^A(x)} -Z^a(x)\,G^a(x)\right\}F\right\rangle, \tag{5.11}$$

in which $\lambda$, an arbitrary parameter of dimension inverse length, has replaced the SIAG bosonizing-kernel $\not{D}\pm m$ of equations (5.1) and (5.3). The naive equations (5.11) form a $\lambda$-family of alternative descriptions of regularized gauge theory with fermions which, however, do not correspond to any known equilibrating stochastic system. The apparent simplicity of the naive equations leads directly to a significant reduction in the number of required SD vertices. A compensating aspect arises, however, in that the corresponding solid line factors pick up matrix structure for the fermions, and are more difficult to handle.

## 6. Background Fields, Anomalies and Currents

Regularized fermions in a background gauge field may be described either a) at the $(d+1)$-dimensional stochastic level, by dropping the $\dot{A}^a_\mu$ equation, along with the Zwanziger terms, in the SIAG-Langevin system (2.4), or b) at the $d$-dimensional level of the SD equations (5.1) by dropping terms with gluonic functional derivatives, along with the Zwanziger terms,

$$0=\int(dx)\left\langle\left\{\left[(\bar{\not{D}}_x-m)^{AB}_{ij}\left(-\frac{\delta S}{\delta\bar{\psi}^B_j(x)}+\int(dy)(\mathbb{R}^2_{xy})^{BC}_{jk}\frac{\delta}{\overline{\psi^C_k(y)}}\right)\right]\frac{\delta}{\delta\psi^A_i(x)}+\left[\left(-\frac{\delta S}{\delta\psi^B_j(x)}+\int(dy)(\mathbb{R}^2_{yx})^{CB}_{kj}\frac{\delta}{\delta\psi^C_k(y)}\right)\cdot(\bar{\not{D}}_x+m)^{BA}_{ji}\right]\frac{\delta}{\delta\bar{\psi}^A_i(x)}\right\}F\right\rangle. \tag{6.1}$$

In either case, the system is essentially a free theory, and the exact fermionic $n$-point functions are easily obtained. Choosing $F=\psi^A_i(x)\,\bar{\psi}^B_j(y)$ in (6.1), for example, we obtain

$$\langle\psi^A_i(x)\,\bar{\psi}^B_j(y)\rangle=[(\{\not{D}+m\}^{-1}\,\mathbb{R}^2)^{AB}_{ij}]_{xy} \tag{6.2}$$

and the $n$-point functions are constructed as usual by Wick's theorem. From this result, or an examination of the corresponding Fokker-Planck equation, it follows that the regularized Euclidean effective action[6]

$$S_{\text{eff}}=\int(dx)(dy)\,\bar{\psi}^A_i(x)[(\{\not{D}+m\}\,\mathbb{R}^{-2})^{AB}_{ij}]_{xy}\,\psi^B_j(y) \tag{6.3}$$

provides an equivalent description of the fermionic equilibrium averages.

The fermionic averages $\langle\psi_1\ldots\psi_n\bar{\psi}_1\ldots\bar{\psi}_m\rangle$ of the model (as well as any finite-derivative composite operator) are successfully regularized in $d$ dimensions by, say, the heat-kernel regulator $\mathbb{R}=\exp(\not{D}^2/\Lambda^2)$, which we adopt in the following discussion. On the other hand, the existence of a well-defined equivalent action formulation for the background field model (which does not exist in the case of regularized dynamical gauge fields [2, 3]) means that the problem of Lee and Zinn-Justin [20] has not been eliminated, and these background field models cannot be totally satisfactory[7]. We will return to these limitations on the background field models after a brief discussion of axial and chiral anomalies, for which the modelling appears to be adequate.

As a simple example, we compute the axial anomaly in four dimensions,

$$\begin{aligned}&\partial^x_\mu\langle\bar{\psi}(x)\,\gamma_5\,\gamma_\mu\,\psi(x)\rangle\\&=\int(dz)\,\mathrm{Tr}[\gamma_5(\not{\partial})_{zy}(\{\not{D}+m\}^{-1}\,\mathbb{R}^2)_{xz}-\gamma_5(\not{\partial})_{xz}(\{\not{D}+m\}^{-1}\,\mathbb{R}^2)_{zy}]|_{y=x} && (6.4\text{a})\\&=-2\,\mathrm{Tr}[\gamma_5(\mathbb{R}^2)_{xx}]-2m\langle\bar{\psi}(x)\,\gamma_5\,\psi(x)\rangle, && (6.4\text{b})\end{aligned}$$

where the trace is over spinor and color indices. To obtain the final line, we have rewritten $\not{\partial}_x=\not{D}_x-ig\not{A}(x)$, and made use of the fact that $\mathbb{R}(\not{D}^2)$ commutes with $\not{D}$. The first term of the result (6.4b) is the anomaly term due to the presence of the regulator. We have used the heat kernel expansion [5] to obtain the standard result $F^a_{\mu\nu}\,\tilde{F}^a_{\mu\nu}(-g^2\,C_R/8\pi^2)$ for the anomaly term at large $\Lambda$.

The Noether structure of this current is easily understood in the effective action formulation (6.3). Consider the *regularized* infinitesimal axial transformation[8]

$$\psi'(x)-\psi(x)=i\int(dy)\,\gamma_5(\mathbb{R}^2)_{xy}\alpha(y)\,\psi(y)+O(\alpha^2) \tag{6.5a}$$

$$\bar{\psi}'(x)-\bar{\psi}(x)=i\int(dy)\,\bar{\psi}(y)\,\alpha(y)(\mathbb{R}^2)_{yx}\,\gamma_5+O(\alpha^2), \tag{6.5b}$$

which possesses a finite Jacobian,

$$\left|\frac{\delta(\bar{\psi}',\psi')}{\delta(\bar{\psi},\psi)}\right|=1+2i\int(dx)\,\alpha(x)\,\mathrm{Tr}[\gamma_5[\mathbb{R}^2(\not{D}^2)]_{xx}]+O(\alpha^2) \tag{6.6}$$

as computed from Eq. (6.5). Following Fujikawa [24], (6.4b) is immediately obtained as a Ward identity. We emphasize that this regularized application of Fujikawa's idea is not formal.

Moreover, for an effective action with regulator dependence $\mathbb{R}^{-r}$, the Noether transformations

[6] A similar but not identical action was obtained in [23] by fifth-time smearing of a background field problem

[7] The Lee and Zinn-Justin problem occurs whenever a well-defined action formulation is available. The problem also occurs in fifth-time smearing of background field problems (see, e.g. footnote 6)

[8] A finite form of this regularized axial transformation is $\psi'(x)=\int(dy)[\exp(i\gamma_5\,\mathbb{R}^2\,\alpha)]_{xy}\,\psi(y)$, $\bar{\psi}'(x)=\int(dy)\,\bar{\psi}(y)[\exp(i\alpha\,\mathbb{R}^2\,\gamma_5)_{yx}]$. With regard to the corresponding vector transformation and Noether current $J_\mu=\bar{\psi}\,\gamma_\mu\,\psi$ we note that the implied Ward identities at finite $\Lambda$ include $\langle\partial\cdot J\rangle=0$, but $\langle\partial\cdot J(x)\,J_\nu(y)\rangle\neq 0$, so that order $\Lambda^2$ terms persist in $\langle J_\mu(x)\,J_\nu(y)\rangle$. This Noether current therefore is not the current to which regularized dynamical gauge fields couple

$\delta\psi(x)=i\gamma_5\int(dy)(dz)(\mathbb{R}^r)_{xy}\,\alpha(y)(\mathbb{R}^{n+1-r})_{yz}\,\psi(z)$, $\delta\bar\psi(x)=i\int(dy)\,\bar\psi(y)\,\alpha(y)(\mathbb{R}^{n+1})_{yx}\,\gamma_5$ with $n\geq 0$ give rise to currents $\int(dy)\,\bar\psi(x)\gamma_5\,\gamma_\mu(\mathbb{R}^{n+1-r})_{xy}\,\psi(y)$ and corresponding anomaly terms $-2\,\mathrm{Tr}[\gamma_5\,\mathbb{R}^{n+1}]$. Each of these currents is regularized and exhibits the same correct anomaly at large $\Lambda$. Note that the case $r=0$ corresponds to regularized (point-splitting) anomaly computations without regularizing the action.

The background field model is also adequate for chiral anomalies. In this case, the regularized $d=2l$ dimensional SIAG-Langevin systems may be taken as

$$(\dot\psi_L)_i^A(x,t)=(\not{D}_x^2\,\psi_L)_i^A(x,t)-(\not{D}_x)_{ij}^{AB}\int(dy)(\mathbb{R}_{jl}^{BC})_{xy}(\eta_+^C)_l(y,t)+\int(dy)(\mathbb{R}_{ij}^{AB})_{xy}(\eta_-^B)_j(y,t),\tag{6.7a}$$

$$(\dot{\bar\psi}_L)_i^A(x,t)=(\bar\psi_L\,\overleftarrow{\not{D}}_x)_i^A(x,t)+\int(dy)(\bar\eta_+^B)_j(y,t)(\mathbb{R}_{ji}^{BA})_{yx}+\int(dy)(\bar\eta_-^C)_l(y,t)(\mathbb{R}_{lj}^{CB})_{yx}(\overleftarrow{\not{D}}_x)_{ji}^{BA},\tag{6.7b}$$

$$\langle(\eta_\pm^A)_i(x,t)(\bar\eta_\pm^B)_j(x',t')\rangle_\eta=\frac{1}{2}\delta^{AB}(1\pm\gamma_{d+1})_{ij}\,\delta(x-x')\,\delta(t-t'),\tag{6.7c}$$

where

$$\psi_L\equiv\frac{1}{2}(1-\gamma_{d+1})\,\psi,\quad \bar\psi_L\equiv\frac{1}{2}\bar\psi(1+\gamma_{d+1}),\tag{6.8a}$$

$$\gamma_{d+1}\equiv i^{d/2}\prod_{\mu=0}^{d-1}\gamma_\mu,\quad \gamma_{d+1}^2=1.\tag{6.8b}$$

Alternatively, the Schwinger-Dyson equations

$$0=\int(dx)\left\langle\left\{(\not{D}^2\,\psi_L)_i^A(x)\frac{\delta}{\delta(\psi_L)_i^A(x)}+(\bar\psi_L\,\overleftarrow{\not{D}}^2)_i^A(x)\frac{\delta}{\delta(\bar\psi_L)_i^A(x)}+\frac{1}{2}\int(dy)\,[\mathbb{R}_{xy}^2\,\overleftarrow{\not{D}}_y-\not{D}_x\,\mathbb{R}_{xy}^2]_{ij}^{AB}\times\frac{\delta}{\delta(\bar\psi_L)_j^B(y)}\frac{\delta}{\delta(\psi_L)_i^A(x)}\right\}F\right\rangle\tag{6.9}$$

may be employed. In either case, the equilibrium fermion averages may be described by the effective action $S_{\rm eff}=\int(dx)(dy)\,\bar\psi_L(x)(\not{D}\,\mathbb{R}^{-2})_{xy}\,\psi_L(y)$. With the heat-kernel regulator $\mathbb{R}=\exp(\not{D}^2/\Lambda^2)$, we have verified the following results

$$\partial_\mu\langle\bar\psi_L(x)\,\gamma_\mu\,\psi_L(x)\rangle=-\mathrm{Tr}\,[\gamma_{d+1}(\mathbb{R}^2)_{xx}]\simeq-g^2\,[(d/2)!(4\pi)^{d/2}]^{-1}\,\varepsilon_{\mu_1\mu_2\ldots\mu_d}\times\mathrm{Tr}\,[F_{\mu_1\mu_2}\,F_{\mu_3\mu_4}\ldots F_{\mu_{d-1}\mu_d}]\tag{6.10a}$$

$$D_\mu^{ab}\langle\bar\psi_L(x)\,\gamma_\mu\,T^b\,\psi_L(x)\rangle=-\mathrm{Tr}\,[T^a\,\gamma_{d+1}(\mathbb{R}^2)_{xx}]\simeq-g^2\,[(d/2)!(4\pi)^{d/2}]^{-1}\,\varepsilon_{\mu_1\mu_2\ldots\mu_d}\times\mathrm{Tr}\,[T^a\,F_{\mu_1\mu_2}\,F_{\mu_3\mu_4}\ldots F_{\mu_{d-1}\mu_d}],\tag{6.10b}$$

where $F_{\mu\nu}\equiv F_{\mu\nu}^a\,T^a$. The result (6.10) is the usual covariant [25] form of the singlet and non-abelian chiral anomalies.

The zero modes may be treated more carefully, either by the method of Egorian, Nissimov and Pacheva [12] at the stochastic level, or directly at the $d$-dimensional level by including an infrared cutoff in the regulator which appears in the SD equations (e.g. $\mathbb{R}=\exp(\not{D}^2/\Lambda^2)-\exp(\not{D}^2/\bar\Lambda^2)\xrightarrow[\bar\Lambda\to 0]{}\exp(\not{D}^2/\Lambda^2)\cdot[1-P_0]$, where $P_0$ is the zero mode projector).

These regularized Noether currents (e.g. $\bar\psi\,\gamma_\mu\,\psi$, $\bar\psi\,\gamma_5\,\gamma_\mu\,\psi$ and the chiral anomalous currents of (6.10) are not the currents to which background gauge fields couple. As seen, for example, in (6.3), the background fields exhibit high-derivative coupling to the fermions, so that, e.g.

$$J_\mu^a[A;x]\equiv\frac{\delta W[A]}{\delta A_\mu^a(x)}=-\int(dy)(dz)\left\langle\bar\psi_i^A(y)\frac{\delta}{\delta A_\mu^a(x)}\cdot[(\{\not{D}+m\}\,\mathbb{R}^{-2})_{ij}^{AB}]_{yz}\,\psi_j^B(z)\right\rangle=\int(dy)(dz)\,\mathrm{Tr}\left[\left(\frac{\delta}{\delta A_\mu^a(x)}\cdot(\{\not{D}+m\}\mathbb{R}^{-2})_{yz}\right)\left(\frac{\mathbb{R}^2}{\not{D}+m}\right)_{zy}\right]\tag{6.11}$$

is not regularized[9]. This is the problem of Lee and Zinn-Justin. It follows that the consistent [25] form of anomalies cannot be studied in these background field models. A deeper consequence is that the gauge-invariance of the background field models is, in this sense, only formal.

The correctly regularized gauge-field amplitudes are only obtained with the fully quantized models of the previous sections, in which the problem of Lee and Zinn-Justin does not occur, and gauge-invariance is not formal.

It would be interesting to compute the axial and chiral anomalies directly from the regularized Green functions of the fully quantized gauge theories. The chiral case is of particular interest, since the fully quantized discussion provides an alternative completely regularized approach to the internal consistency of such theories [26, 27]. In this connection, we note that the chiral anomalous currents of (6.10) are not a priori the currents to which regularized dynamical chiral gauge fields couple.

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[9] The statement for $W[A]=\sum_n W_n\,A^n$ in $d$-dimensions is that, independent of $\mathbb{R}(\not{D}^2)$, $W_n$ is not regularized for $n\leq d$

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