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Non-Grassmann Formulation of Regularized Gauge Theory with Fermions [1]

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Abstract. Regularized continuum gauge theory coupled to quadratic matter simplifies significantly on integration of the matter fields. As an illustration, we discuss in some detail the resulting non-Grassmann formulation of regularized gauge theory with Dirac fermions.

1. Introduction

Principles adequate for presumably non-perturbative continuum regularization of any quantum field theory have recently been given [1–6], and the scalar prototype [2], scalar electrodynamics [3], gauge theory [3–5] and gauge theory with fermions [6] have been studied in detail. The ingredients are Langevin equations [7] or, equivalently, Schwinger-Dyson systems, with appropriate covariant-derivative regularization. Although the principles of the regularization scheme remain constant, the study of stochastic and/or Schwinger-Dyson systems is in its infancy. In some cases, therefore, it is inevitable that physically equivalent but simpler formulations will be found.

In this paper, we report that integration of the matter fields in regularized gauge theory gives such a simplification. In particular, we discuss in some detail the resulting non-Grassmann formulation of regularized gauge theory with Dirac fermions, analogues of which have been studied on the lattice [8]. The integrated regularized formulation of scalar electrodynamics is also given. In general, considerable simplification is found in the weak-coupling expansion of these integrated systems.

The plan of the paper is as follows. In Sect. 2 we state and discuss the integrated regularized formulation for gauge theory with Dirac fermions, and for scalar electrodynamics. The approach is basically Schwinger-Dyson, though stochastic equivalents are also given. Section 3 discusses the regularized Schwinger-Dyson rules for the weak-coupling expansion of the integrated systems in d-dimensions, emphasizing the simplicity of the rules relative to the regularized SIAG systems [6]. Section 4 illustrates the simplicity of the rules in a computation of the fermionic contribution to the leading term in the vacuum polarization. The result is transverse in any dimension, in contrast to the Zwanziger [9]-SIAG [10] effect found in [11, 6].

In Sect. 5, we begin to study the relation of the integrated systems to the regularized Grassmann SIAG systems [6]. A λ-family of regularized SIAG systems is given, with the case $\lambda=1$ corresponding to the systems of [6]. Section 6 discusses some indications that the large λ limit of this SIAG λ-family is the integrated formulation. A proof of this correspondence is given in Sect. 7. The techniques introduced in this section are equivalent to *regularized integration* of matter fields *within* the regularized theories. Finally, Sect. 8 discusses the regularized integration technique in a number of other cases, including scalar electrodynamics.

2. Regularized Fermions without Grassmann Variables

In this section, we state the basic Schwinger-Dyson formulation of the new regularization scheme for integrated gauge-theory fermions.

We wish to regularize the d-dimensional gauge theory with Dirac fermions whose Euclidean action is

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$$S=S_{\rm YM}+\int (dx)\,\bar{\psi}_i^A(\vec{\not{D}}_{ij}^{AB}+\delta^{AB}\delta_{ij}m)\psi_j^B, \tag{2.1}$$

where $S_{\rm YM}$ is the usual Euclidean Yang-Mills action, and our fermionic notation follows [6]. We propose the regularized Schwinger-Dyson (SD) equations

$$0=\left\langle\left[L_{\rm YM}+ig\int (dx)\,{\rm Tr}[T^a\gamma_\mu\{(\not{D}+m)^{-1}\mathbb{R}^2\}_{xx}]\cdot\frac{\delta}{\delta A_\mu^a(x)}\right]F[A]\right\rangle \tag{2.2}$$

for computing the averages of any functional $F[A]$ of the gauge field. Here

$$L_{\rm YM}\equiv -\int (dx)\left[\frac{\delta S_{\rm YM}}{\delta A_\mu^a(x)}+Z^b(x)D_\mu^{ba}\right]\frac{\delta}{\delta A_\mu^a(x)}+\Delta \tag{2.3}$$

is the usual regularized SD operator for pure Yang-Mills theory, with regularized functional Laplacian [3, 5]

$$\Delta=\int (dx)(dy)(R^2)_{xy}^{ab}\frac{\delta^2}{\delta A_\mu^a(x)\,\delta A_\mu^b(y)} \tag{2.4}$$

as a functional of the gauge-field regulator $R(\Delta)$. The fermionic regulator $\mathbb{R}(\not{D}^2)$ appears in the additional *determinantal term* of (2.2), which generates the regularized fermionic contributions to any gauge-field average.

Our prescription for the fermionic averages is

$$\begin{aligned}&\langle\exp[\int (dx)\{\bar{\chi}_i^A(x)\psi_i^A(x)+\bar{\psi}_i^A(x)\chi_i^A(x)\}]\,F[A]\rangle\\&=\langle\exp[\int (dx)(dy)\,\bar{\chi}_i^A(x)[(\{\not{D}+m\}^{-1}\mathbb{R}^2)_{ij}^{AB}]_{xy}\cdot\chi_j^B(y)]\,F[A]\rangle,\end{aligned} \tag{2.5}$$

where χ_i^A, $\bar{\chi}_j^B$ are Grassmann sources. This is a shorthand* which records the vanishing of Green functions with non-zero fermion number, along with the family of statements

$$\begin{aligned}&\langle\psi_i^A(x)\bar{\psi}_j^B(y)F_1[A]\rangle\\&=\langle[(\{\not{D}+m\}^{-1}\mathbb{R}^2)_{ij}^{AB}]_{xy}F_1[A]\rangle,\end{aligned} \tag{2.6a}$$

$$\begin{aligned}&\langle\psi_i^A(x)\psi_j^B(y)\bar{\psi}_k^C(u)\bar{\psi}_l^D(v)F_2[A]\rangle\\&=\langle[(\{\not{D}+m\}^{-1}\mathbb{R}^2)_{il}^{AD}]_{xv}\cdot[(\{\not{D}+m\}^{-1}\mathbb{R}^2)_{jk}^{BC}]_{yu}F_2[A]\rangle\\&-\langle[(\{\not{D}+m\}^{-1}\mathbb{R}^2)_{ik}^{AC}]_{xu}\cdot[(\{\not{D}+m\}^{-1}\mathbb{R}^2)_{jl}^{BD}]_{yv}F_2[A]\rangle,\end{aligned} \tag{2.6b}$$

and so on with arbitrary $F_n[A]$, where n labels fermion pair number. The fermionic prescription (2.5) expresses all fermionic averages in terms of gauge-field averages, which may then be computed from the SD equations (2.2).

It is easy to check the validity of the SD system (2.2, 5) as a regularization of the theory whose formal action is (2.1). When $\mathbb{R}=R=1$ and Zwanziger's Z^a is omitted, the SD equations (2.2) are equivalent to the formal relations

$$0=\int \mathscr{D}A\int (dx)\frac{\delta}{\delta A_\mu^a(x)}\left[\det[\not{D}+m]\,e^{-S_{\rm YM}}\frac{\delta}{\delta A_\mu^a(x)}F[A]\right] \tag{2.7}$$

at the action level: The determinantal term in (2.2) is simply a version of the formal expression $\delta{\rm Tr}[\ln(\not{D}+m)]/\delta A_\mu^a$, regularized after the differentiation. Similarly, the fermionic prescription (2.5) is a regularized version of standard unregularized formal relations.

We also remark that any fermionic object gauge-invariant in A, ψ, and $\bar{\psi}$ is given, according to the prescription (2.5), by a gauge-invariant construction in A. Since the SD operator without the Zwanziger gauge-fixing term is gauge-invariant, it follows [3] that *all* gauge-invariant observables $F_{\rm GI}[A]$, including the fermionic constructions, will be independent of the Zwanziger gauge-fixing.

The "quenched" approximation, in which internal fermions are suppressed, is also easy to write down. One simply omits the determinantal term in the SD equations (2.2), while maintaining the fermionic prescription (2.5).

We further mention stochastic equivalent prescriptions. Following [5], the SD equations (2.2) are equivalent to

$$\begin{aligned}\dot{A}_\mu^a(x,t)=&-\frac{\delta S_{\rm YM}}{\delta A_\mu^a}(x,t)+D_\mu^{ab}Z^b(x,t)\\&+\int (dy)\,R_{xy}^{ab}\,\eta_\mu^b(y,t)\\&+ig\,{\rm Tr}[T^a\gamma_\mu\{(\not{D}+m)^{-1}\mathbb{R}^2\}_{xx}(t)]\\&-\int (dy)(dz)\,R_{yz}^{bc}\frac{\delta R_{xy}^{ab}}{\delta A_\mu^c(z)},\end{aligned} \tag{2.8a}$$

$$\langle\eta_\mu^a(x,t)\,\eta_\nu^b(x',t')\rangle=2\delta^{ab}\,\delta_{\mu\nu}\,\delta(x-x')\,\delta(t-t'), \tag{2.8b}$$

with the Stratonovich calculus [12]. In this case, both the determinantal term and the final RVC_1 counterterm [5] correspond to loops in the Langevin trees. Alternatively, the RVC_1 counterterm may be omitted with the Ito calculus [12], which simply eliminates all functional derivatives of R (all RVC_1's).

The case of fermions illustrates the steps to follow

* The shorthand (2.5) also includes the usual [2, 3, 6] global sign convention, implied by the SD boundary condition that the averages have the usual permutation symmetries and antisymmetries

in integrating out any quadratic matter fields coupled to the gauge field. As a further example, we leave as an exercise for the reader to verify that the SD equations

$$0=\int(dx)\left\langle\left[-\frac{\delta S^{(0)}}{\delta A_\mu(x)}+\partial_\mu Z(x)+\int(dy)\,R^2_{xy}(\Box)\frac{\delta}{\delta A_\mu(y)}\right.\right.$$
$$\left.\left.+ie\{\Delta^{-1}R^2(\Delta),D_\mu\}_{xx}\right]\frac{\delta}{\delta A_\mu(x)}F[A]\right\rangle \tag{2.9}$$

describe regularized scalar electrodynamics [3] after integration of the charged scalars. Here

$$S^{(0)}=\tfrac{1}{4}\int(dx)\,F_{\mu\nu}F_{\mu\nu},\quad \Delta\equiv D_\mu D_\mu=(\partial_\mu-ieA_\mu)^2, \tag{2.10}$$

and the curly bracket in the (last) determinantal term denotes anticommutator. Similarly, with J a complex source, the prescription

$$\langle\exp[\int(dx)\{J^*(x)\,\phi(x)+\phi^*(x)\,J(x)\}]\,F[A]\rangle$$
$$=\langle\exp[-\int(dx)(dy)\,J^*(x)[\Delta^{-1}R^2(\Delta)]_{xy}J(y)]\,F[A]\rangle \tag{2.11}$$

gives the charged scalar averages.

3. Regularized Schwinger-Dyson Diagrams

In this section, we give the regularized SD rules for the weak-coupling expansion of the integrated SD systems (2.2, 5). For simplicity, we choose the canonical Zwanziger term $Z^a=\alpha^{-1}\partial\cdot A^a$, and heat kernel regularization [5]

$$R=\exp(\Delta/\Lambda^2),\qquad \mathbb{R}=\exp(\not{D}^2/\Lambda^2), \tag{3.1}$$

for which the systems are regularized* to all orders in d-dimensions.

Since the fermionic prescription (2.5) reduces fermionic averages to gauge-field averages, we concentrate first on the gauge-field SD rules which follow from the SD equations (2.2). The pure Yang-Mills SD rules for the operator L_{YM} have been discussed in [3, 5], so we discuss here only the additional rules necessary to include the contributions of the determinantal term to the gauge-field averages.

Solid line factors remain purely gluonic [3], but extra *composite SD vertices* (fermion loops) are generated by the determinantal term. Since this term has one hanging derivative, the (non-local) coefficient of each power of the gauge field A in the weak-coupling expansion of $\text{Tr}[T^a\gamma_\mu\{(\not{D}+m)^{-1}\mathbb{R}^2\}_{xx}]$ corresponds to a SD composite vertex with one incoming gluon line.

Study of the composite SD vertices begins with

* The minimal SD regulator for $d=4$ dimensions (QCD) are $R=(1-\Delta/\Lambda^2)^{-1}$, $\mathbb{R}=(1-\not{D}^2/\Lambda^2)^{-1}$, as in [6]

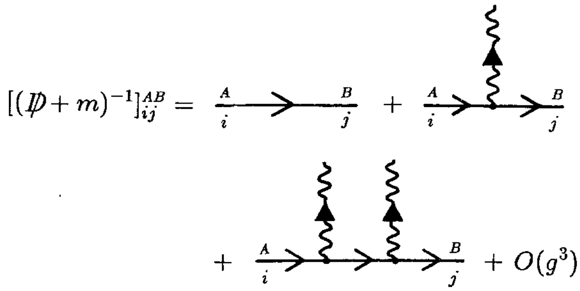

Fig. 1. Fermion propagator strings

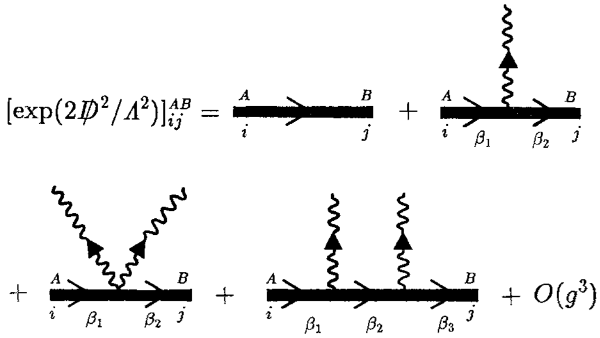

Fig. 2. Fermion heat kernel as fermion regulator strings

the expansion of the fermion propagator

$$(\not{D}+m)^{-1}=\sum_{n=0}^{\infty}[(\not{\partial}+m)^{-1}\{-ig\,T^a\not{A}^a\}]^n(\not{\partial}+m)^{-1}, \tag{3.2a}$$

and the square of the fermion regulator

$$\mathbb{R}^2=\exp(2\not{D}^2/\Lambda^2)=\exp(2\Box/\Lambda^2)$$
$$+\sum_{n=2}^{\infty}\int_0^1\prod_{j=1}^{n}d\beta_j\,\delta\left(1-\sum_{k=1}^{n}\beta_k\right)\exp(2\beta_1\Box/\Lambda^2)\,V$$
$$\cdot\exp(2\beta_2\Box/\Lambda^2)\,V\ldots V\exp(2\beta_n\Box/\Lambda^2), \tag{3.2b}$$

$$[V^{AB}_{ij}]_{xy}\equiv\frac{2}{\Lambda^2}[g\Gamma^{(1)}(x)+g^2\Gamma^{(2)}(x)]^{AB}_{ij}\,\delta(x-y), \tag{3.2c}$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are the one- and two-gluon regulator vertices defined in [6]. Figures 1 and 2 give the diagrammatic interpretation of the expansions (3.2a) and (3.2b) as, respectively, *fermion propagator strings* and *fermion regulator strings*. The product $(\not{D}+m)^{-1}\mathbb{R}^2$ of the two expansions, as a sum of *regularized fermion propagator strings* (*RFP strings*), is shown in Fig. 3.

The β-parameters in any RFP string are integrated from zero to one, subject to the constraint that their sum is unity [5]. The thin arrow (—>—) on the

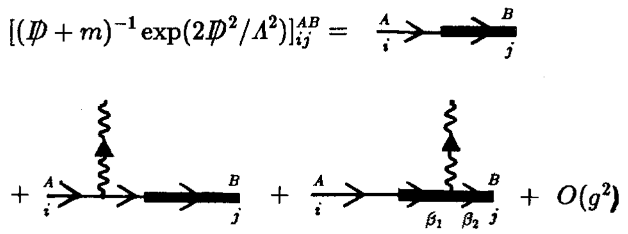

Fig. 3. Regularized fermion propagator strings (RFP strings)

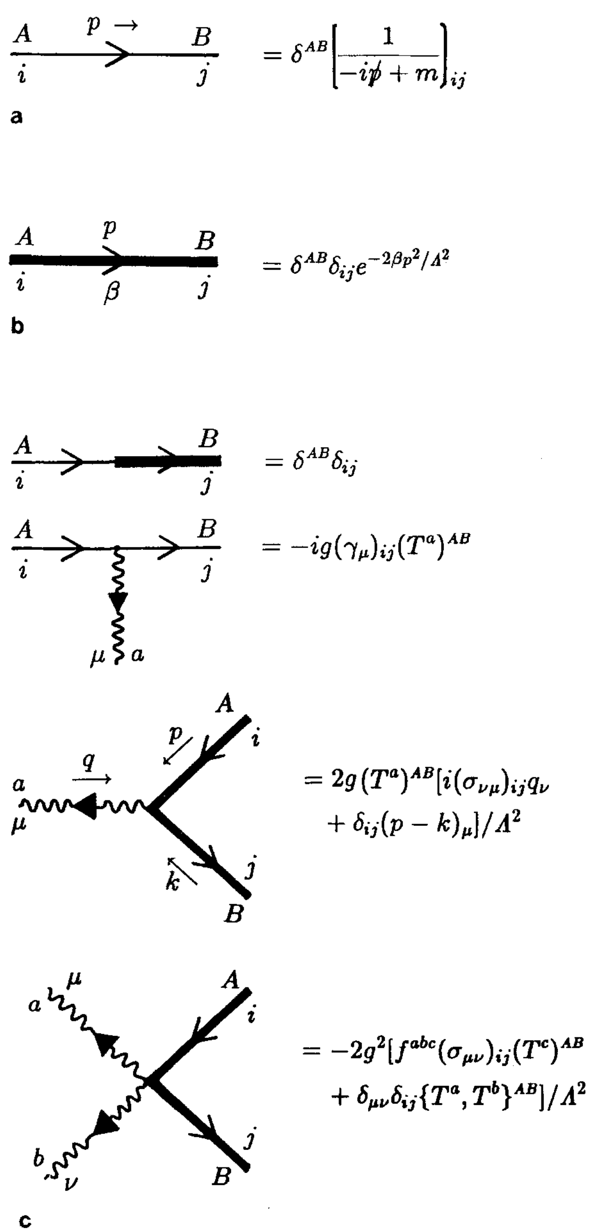

Fig. 4a–c. Diagrammatic rules for RFP string construction. **a** Fermion propagator. **b** Fermion regulator propagator. **c** RFP vertices

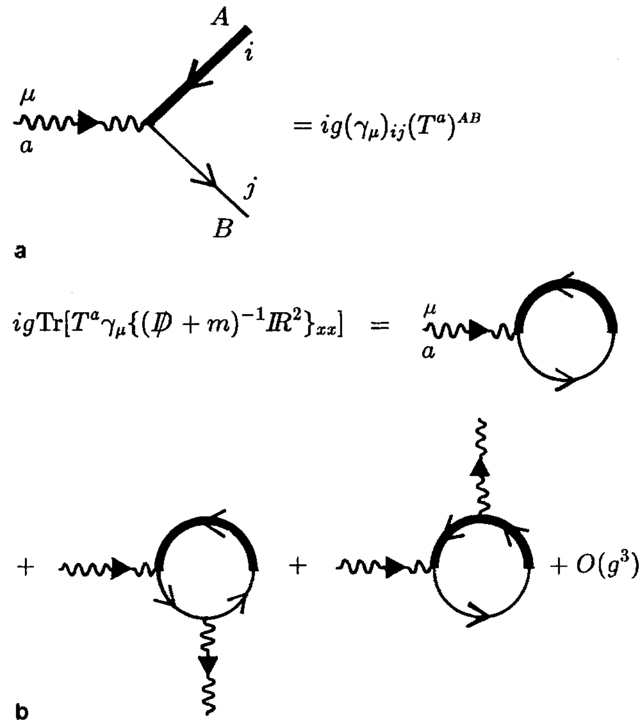

Fig. 5a, b. Expansion of RFP loops. **a** Determinantal vertex. **b** RFP loops

fermion lines in these diagrams indicate the charge flow, while the solid arrows (—▶—) on the gluon lines track the direction of allowed (gauge field) SD ordering (or, equivalently, the direction of decreasing Markov time in the Langevin formulation). Figure 4 gives the momentum-space diagrammatic rules for constructing the RFP strings in terms of *RFP vertices.*

The composite SD vertices are simply RFP loops (Fig. 5b), *closed* with a *determinantal vertex* $igT^a\gamma_\mu$, shown in Fig. 5a. Each RFP loop is a SD *vertex unit* [3], requiring *no SD ordering of the RFP vertices within a loop.* This feature results in a general reduction of the number of SD pictures for each SD diagram relative to the regularized SIAG systems [6]. We also note that the first loop (tadpole) in the expansion of Fig. 5b is zero since $\mathrm{Tr}[\gamma_\mu T^a]=0$, so internal fermions begin to contribute at order g^2. With the rules for L_{YM} in [3, 5], this completes the SD rules for the computation of any gauge field average.

Counting the determinantal vertex with the RFP vertices of Fig. 4, we have introduced five vertices in total. This should be compared with the regularized SIAG systems [6], in which nine vertices were necessary. As we shall see below, this results in a serious reduction of the number of diagrams at a given order in weak coupling.

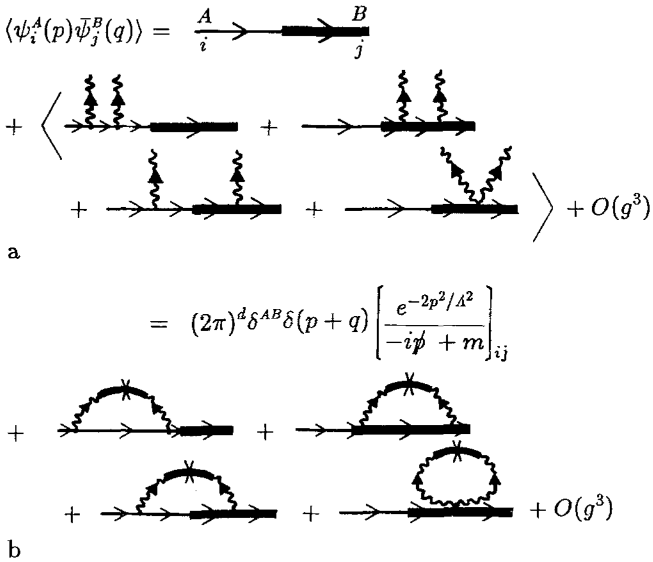

Fig. 6a, b. Diagrams for $\langle \psi_i^A(p)\bar{\psi}_j^B(q)\rangle$. **a** is the RFP string expansion and **b** shows the SD gauge-field averages

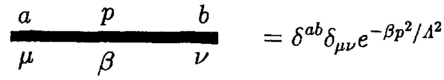

Fig. 7. Gauge-field regulator propagator

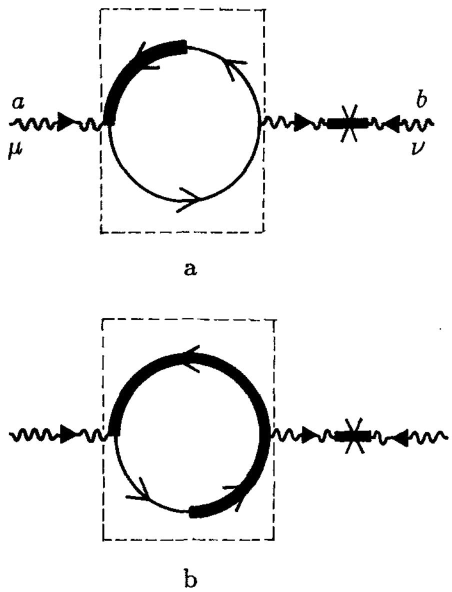

Fig. 8a, b. Second order vacuum polarization diagrams

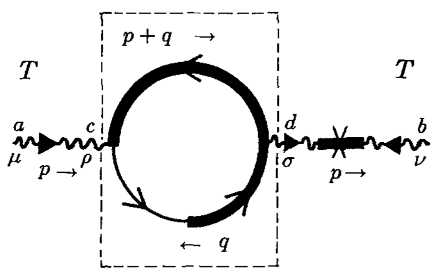

Fig. 9. Diagram 8b with indices; the external gluons are transverse

Finally, to compute the fermionic averages, we expand each factor of $(\not{D}+m)^{-1}\mathbb{R}^2$ in the prescription (2.5) as *open* (in general) RFP strings, and then use the gauge-field SD rules above to compute the resulting gauge-field averages. As an illustration, we exhibit the diagrams for $\langle \psi_i^A(p)\bar{\psi}_j^B(q)\rangle$ through order g^2 in Fig. 6. In this figure, we have adopted a thick line without an arrow (Fig. 7) as the heat-kernel gauge-field regulator propagator [5]. Figure 6a shows the expansion of the two-point fermionic construction as open RFP strings, while Fig. 6b shows the subsequent gauge-field averaging, according to the gauge-field SD rules above. For example, the value

$$-g^2(2\pi)^d\,\delta(p+q)\,e^{-2p^2/\Lambda^2}(T^a T^a)^{AB} \cdot \int (dk)\left[\frac{T_{\mu\nu}(k)+\alpha L_{\mu\nu}(k)}{k^2}\right]e^{-2k^2/\Lambda^2} \cdot\left[\frac{1}{-i\not{p}+m}\gamma_\mu\frac{1}{-i\not{p}+i\not{k}+m}\gamma_\nu\frac{1}{-i\not{p}+m}\right]_{ij} \quad (3.3)$$

is obtained for the first loop diagram of Fig. 6b.

4. Fermionic Contribution to the Vacuum Polarization

As an illustration of the simplicity of the integrated formulation, and as an explicit check on the gauge-invariance of the regularization, we use the SD rules of the previous section to compute the leading fermionic contribution to the gauge-field vacuum polarization in d-dimensions. In particular, since the regularized Yang-Mills contribution to the gluon mass is zero [1, 3, 5], the fermionic mass contribution must also vanish.

There are altogether $2\times 2=4$ diagrams with one internal fermion loop which contribute to the vacuum polarization $\Pi_{\mu\nu}^{ab}(p)$. Two of these are shown in Fig. 8, while the other two are trivially obtained from those of Fig. 8 by interchanging (a, μ, p) with $(b, \nu, -p)$. The dotted box [3, 6] in each diagram indicates here that the entire RFP loop is treated as a vertex unit. The small number of diagrams is noteworthy, since the regularized SIAG formulation [6] required 24 diagrams at this level.

As an explicit example, we discuss the evaluation of the SD diagram 8b, which contains a fermionic

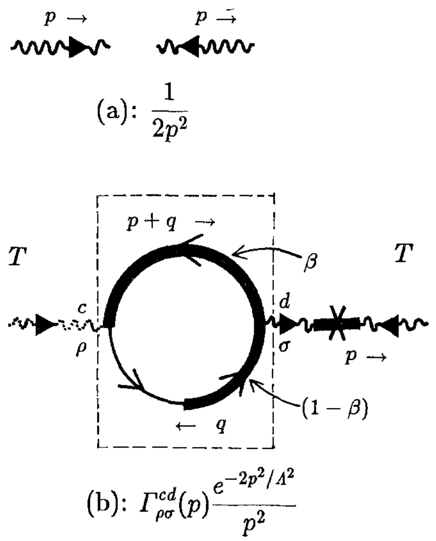

Fig. 10a, b. SD pictures for Fig. 9

regulator vertex. The case with both external gluons transverse is shown with all relevant indices in Fig. 9, and the sequence of SD pictures for the diagram is given in Fig. 10. The RFP-loop factor $\Gamma^{cd}_{\rho\sigma}(p)$ of Fig. 10 is readily computed from the rules of Fig. 4,

$$\Gamma^{cd}_{\rho\sigma}(p)=\int_0^1 d\beta \int (dq)\, e^{-2\beta(p+q)^2/\Lambda^2}\, e^{-2(1-\beta)q^2/\Lambda^2} \cdot ig\,\mathrm{Tr}\left[\left\{T^c\gamma_\rho \frac{1}{-i\not{q}+m}\right\}\frac{2igT^d}{\Lambda^2} \cdot\{-i\sigma_{\nu\sigma}p_\nu-(2q+p)_\sigma\}\right]. \tag{4.1}$$

Collecting the factors from the SD pictures, transverse projectors for gluon lines, and appropriate sums in color, flavor, and Dirac indices, we find

$$-\frac{g^2}{\Lambda^2}C_R d_\gamma \delta^{ab} f \frac{e^{-2p^2/\Lambda^2}}{p^4}\int_0^1 d\beta \cdot \int (dq)\, e^{-2\beta(p+q)^2/\Lambda^2}\, e^{-2(1-\beta)q^2/\Lambda^2}\frac{1}{q^2+m^2} \cdot[2q_\rho q_\sigma+p_\rho q_\sigma+p_\sigma q_\rho-(p\cdot q)\delta_{\rho\sigma}]\,T_{\mu\rho}(p)\,T_{\nu\sigma}(p), \tag{4.2}$$

as the value of the SD diagram in Fig. 9. Here C_R is the Dynkin index of the fermion representation R, f is the number of flavors and $d_\gamma=2^{[d/2]}$ is the dimension of Dirac matrices in d-dimensions.

Diagram 8b with external longitudinal gluons is similarly evaluated. After truncation [3, 6], we find the total contribution of diagram 8b (plus its interchange) to the zero momentum vacuum polarization

$$\Pi^{ab}_{\mu\nu}(0)|_{8b}=\frac{2}{d}\,\delta_{\mu\nu}\,\mathcal{M}^{ab}\Lambda^{d-2}\int_0^\infty dy\,\frac{y^{d/2}\,e^{-2y}}{y+m^2/\Lambda^2}, \tag{4.3}$$

where

$$\mathcal{M}^{ab}\equiv\frac{g^2 C_R d_\gamma \delta^{ab} f}{(4\pi)^{d/2}\Gamma(d/2)}. \tag{4.4}$$

The same contribution with opposite sign is obtained from the ordinary diagram 8a (plus its interchange), so the gluon remains massless to this order in all dimensions.

The leading p^2 contribution to $\Pi^{ab}_{\mu\nu}(p)$ may also be computed by differentiation with respect to external momentum. Adding the contribution of diagram 8a and 8b, together with the $(\mu, a, p)\leftrightarrow(\nu, b, -p)$ interchanges, we obtain the total fermionic contribution to the vacuum polarization in d-dimensions*

$$\Pi^{(f)ab}_{\mu\nu}(p)=-\left(\frac{d-2}{6}\right)\mathcal{M}^{ab}p^2\,T_{\mu\nu}(p)\,\Lambda^{d-4} \cdot\int_0^\infty dy\,\frac{y^{(d-4)/2}\,e^{-2y}}{y+m^2/\Lambda^2}+O(p^4). \tag{4.5}$$

We remark that the result (4.5) is transverse in all dimensions. This is not surprising here, since the fermionic-Zwanziger terms, which cause the fermionic non-transversality phenomenon [11, 6], are not present in the integrated formulation. We shall return to this point below. The standard result $-(g^2 C_R \delta^{ab} f/12\pi^2)p^2\,T_{\mu\nu}\ln(\Lambda^2/m^2)$ is easily obtained from (4.5) when $d=4$.

5. A λ-Family of Regularized SIAG Systems

Our direction in this and the following sections is to obtain the connection between the regularized SIAG systems of [6] and the new integrated formulation of regularized gauge theory with fermions given above. As we shall see in Sect. 7, the connection involves integration of matter fields *within* the regularization scheme.

We begin with the λ-family of regularized SIAG equations

$$\dot{A}^a_\mu(x,t)=-\frac{\delta S}{\delta A^a_\mu}(x,t)+D^{ab}_\mu Z^b(x,t)+\int(dy)\,R^{ab}_{xy}\,\eta^b_\mu(y,t), \tag{5.1a}$$

$$\begin{aligned}\dot{\psi}^A_i(x,t)=&\lambda(\overrightarrow{\not{D}}{}^2_x-m^2)^{AB}_{ij}\psi^B_j(x,t)+\int(dy)(\mathbb{R}^{ab}_{ij})_{xy}(\eta^B_2)_j(y,t)\\&-\lambda(\overrightarrow{\not{D}}_x-m)^{AB}_{ij}\int(dy)(\mathbb{R}^{BC}_{jl})_{xy}(\eta^C_1)_l(y,t)\\&-igZ^a(T^a)^{AB}\psi^B_i(x,t),\end{aligned} \tag{5.1b}$$

* The regular vertex diagram 8b (or its interchange) contributes to renormalization (terms $O(p^2)\times$(growing with Λ)) only when $d>4$

$$\dot{\bar\psi}_i^A(x,t)=\lambda\bar\psi_j^B(x,t)(\overleftarrow{\not{\partial}}_x^2-m^2)_{ji}^{BA}+\int(dy)(\bar\eta_1^B)_j(y,t)(\mathbb{R}_{ji}^{BA})_{yx} +\lambda\int(dy)(\bar\eta_2^C)_l(y,t)(\mathbb{R}_{lj}^{CB})_{yx}(\overleftarrow{\not{\partial}}_x+m)_{ji}^{BA} +igZ^a(T^a)^{BA}\bar\psi_i^B(x,t), \qquad (5.1\,\mathrm{c})$$

plus the usual Gaussian noise averages [6] and Ito calculus. Our primary interest, however, will be in the equivalent λ-family of SD equations

$$0=\Big\langle\Big\{\Delta+\int(dx)\Big(\Big[-\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)}-ig\bar\psi_i^A(x)(T^a)^{AB}(\gamma_\mu)_{ij}\psi_j^B(x)\Big]\frac{\delta}{\delta A_\mu^a(x)} -\lambda\Big[(\overrightarrow{\not{\partial}}_x-m)_{ij}^{AB}\Big(-\frac{\delta S}{\delta\bar\psi_j^B(x)}+\int(dy)(\mathbb{R}_{xy}^2)_{jk}^{BC}\frac{\delta}{\bar\psi_k^C(y)}\Big)\Big]\frac{\delta}{\delta\psi_i^A(x)} -\lambda\Big[\Big(-\frac{\delta S}{\delta\psi_j^B(x)}+\int(dy)(\mathbb{R}_{yx}^2)_{kj}^{CB}\frac{\delta}{\delta\psi_k^C(y)}\Big)(\overleftarrow{\not{\partial}}_x+m)_{ji}^{BA}\Big]\frac{\delta}{\delta\bar\psi_i^A(x)} -Z^a(x)G^a(x)\Big)\Big\}F[A,\psi,\bar\psi]\Big\rangle, \qquad (5.2)$$

where

$$G^a(x)\equiv D_\mu^{ab}\frac{\delta}{\delta A_\mu^b(x)}+ig(T^a)^{AB}\psi_i^B(x)\frac{\delta}{\psi_i^A(x)} -ig(T^a)^{BA}\bar\psi_i^B(x)\frac{\delta}{\delta\bar\psi_i^A(x)} \qquad (5.3)$$

is the generator of non-abelian gauge transformations.

The only difference between (5.1, 2) and the regularized SIAG systems of [6] is the insertion of an extra factor of the dimensionless constant $\lambda>0$ in the SIAG kernel $(\not{\partial}\pm m)$. Such a modification of the kernel does not affect the equilibrium theory at the level of the formal unregularized system, as seen in an examination of the (formal) Fokker-Planck equations without Zwanziger gauge-fixing. Alternatively, when Zwanziger gauge-fixing and the regulators are removed, the SD family (5.2) corresponds to the λ-family of formal statements

$$0=\int\mathscr{D}A\mathscr{D}\bar\psi\mathscr{D}\psi\int(dx)\frac{\delta}{\delta A_\mu^a(x)}\Big[e^{-S}\frac{\delta}{\delta A_\mu^a(x)}F\Big], \qquad (5.4\,\mathrm{a})$$

$$0=-\int\mathscr{D}A\mathscr{D}\bar\psi\mathscr{D}\psi \cdot\int(dx)\Big[\lambda(\overrightarrow{\not{\partial}}_x-m)_{ij}^{AB}\frac{\delta}{\delta\bar\psi_j^B(x)}\Big]\Big[e^{-S}\frac{\delta}{\delta\psi_i^A(x)}F\Big], \qquad (5.4\,\mathrm{b})$$

$$0=-\int\mathscr{D}A\mathscr{D}\bar\psi\mathscr{D}\psi \cdot\int(dx)\frac{\delta}{\delta\psi_j^B(x)}\lambda(\overleftarrow{\not{\partial}}_x+m)_{ji}^{BA}\Big[e^{-S}\frac{\delta}{\delta\bar\psi_i^A(x)}F\Big]. \qquad (5.4\,\mathrm{c})$$

at the action level.

On the other hand, in the presence of the regulator (and/or the Zwanziger gauge-fixing), (5.2) defines a λ-family of *distinct* systems, each of which is a gauge-fixed regularization of the formal theory. As we shall argue in Sect. 7, the large λ limit is the simplest of the family, and in fact corresponds to the integrated formulation above.

The introduction of the parameter λ in (5.2) leads to a few systematic changes, wherever the SIAG kernel enters, relative to the $\lambda=1$ SD rules of [6].

λ-Modified SD Rules

(1) *Solid Line Factors.* Every fermionic term $\lambda(p^2+m^2)$ in a solid line factor now carries an extra factor λ.

(2) *Ordinary Vertices.* All fermionic vertices which originate from $\delta S/\delta\psi$ or $\delta S/\delta\bar\psi$ now carry an extra factor of λ. This includes the second, third and fourth ordinary SIAG vertices in Fig. 1c of [6], except for their α^{-1} fermionic-Zwanziger terms. We may therefore assign an overall λ factor to these vertices, while modifying the fermionic-Zwanziger parameter $\alpha^{-1}\to(\lambda\alpha)^{-1}$. At least formally, this implies the vanishing of the fermionic Zwanziger contributions at large λ (and $\alpha\neq0$) relative to other contributions. We shall return to this below.

(3) *Joining Vertices.* The second and third joining vertices in Fig. 1d of [6] now carry an extra factor λ.

(4) *Contraction.* The fermionic simple contraction factor $\delta^{ab}\delta_{ij}\mathbb{R}_0^2(p)/[2\lambda(p^2+m^2)]$ now carries an additional factor λ^{-1}.

We also note that there are no changes in the regulators or their strings when $\lambda\neq1$.

6. Indications of a Large λ Correspondence

We now employ this λ-family of regularized SD rules to discuss early indications that the regularized SIAG λ-family (5.2) approaches our new integrated regularization as $\lambda \to \infty$. It is trivial to check directly that the resulting regularized λ-tree graphs are independent of λ, and in agreement with the prescription (2.5).

A non-trivial indication of the large λ correspondence comes from a study of the one-loop fermionic contribution to the gluon vacuum polarization in four dimensions. Following the computation of [6], the result for the leading term at arbitrary λ is

$$\Pi^{(f)ab}_{\mu\nu}(p) = \frac{-g^2 C_R \delta^{ab} f}{16\pi^2} p^2 \left[\frac{4}{3} T_{\mu\nu}(p) + \frac{1}{\alpha\lambda} L_{\mu\nu}(p)\right] \cdot \ln \frac{\Lambda^2}{m^2} + \text{terms finite as } \Lambda \to \infty, \tag{6.1}$$

or

$$\langle A^a_\mu(p) A^b_\nu(q)\rangle^{(f)} = (2\pi)^4 \delta(p+q) \frac{-g^2 C_R \delta^{ab} f}{16\pi^2} \cdot \frac{1}{p^2}\left(\frac{4}{3} T_{\mu\nu} + \frac{\alpha}{\lambda} L_{\mu\nu}\right) \ln \frac{\Lambda^2}{m^2} + \text{terms finite as } \Lambda \to \infty. \tag{6.2}$$

We note, in particular, the smooth approach at large λ to the conventional transverse result, in agreement with the result (4.5) of the integrated regularization: The vanishing of the fermionic-Zwanziger contributions at large λ, anticipated in the λ-modified SD rule (2) above, is not formal.

A further indication of the correspondence is found on examination of the 24 SIAG diagrams of this computation at arbitrary external momentum. After cancellation of λ-factors from fermionic vertices against λ^{-1} factors from large λ fermionic solid line factors, we find that the four diagrams of Fig. 4 in [6] vanish at large λ, while the other twenty diagrams add to exactly the four diagrams of the integrated scheme.

7. Integration of Regularized Quadratic Matter at Large λ

In this section, we prove that the SIAG λ-family (5.2) is equivalent at large λ to our new integrated regularization (2.2, 5).

The proof involves two stages. a) A *no-growth theorem*, that all regularized Green functions are bounded by constants at large λ. This part of the proof is given to all orders in weak coupling, though a non-perturbative proof would be preferable. b) A direct *non-perturbative solution of the SD equations* (5.2) *at large* λ, assuming the no-growth theorem. The techniques of this procedure are equivalent to the integration of matter fields within the regularized formulations.

No-Growth Theorem

The intuitive basis for the no-growth theorem is that the unregularized and un-gauge-fixed equilibrium theory is independent of λ, and so exhibits no growth. Dimensional regularization (and Faddeev-Popov gauge-fixing) in SD equations analogous to (5.2), for example, would give λ-independent results for the *sums* of SD diagrams at a given order in weak coupling. Since our regulator reproduces dimensional regularization in weak coupling, we should obtain λ-independence at least at large cutoff Λ, except possibly for Zwanziger effects. In fact, the no-growth theorem below is independent of R and $\mathbb{R}$, and no difficulties are found with Zwanziger's gauge-fixing.

The proof of the no-growth theorem follows immediately from λ-power counting of the SD diagrams. We recall from the λ-SD rules of Sect. 5 that there are only two possible sources of growth in λ. These are the ordinary fermionic vertices of rule (2), and the joining vertices of rule (3), each of which carries a factor λ. In both cases, compensating factors of λ^{-1} are easily found: When a SD picture contains a rule (2)-vertex, the preceding SD picture contains a fermionic solid line factor, which carries an explicit λ^{-1} at large λ. Further, if a SD picture contains a rule (3)-joining vertex, that picture also contains (after some length of regulator string) a rule (4)-contraction factor, with an explicit λ^{-1}. The total power of λ in any SD diagram is therefore never positive, so no growth is possible at large λ.

Solution of SD Equations at Large λ

The first step in this stage is to record the relation implied by the SD λ-family (5.2) for any functional $F[A]$ of the gauge field alone

$$0 = \left\langle \left[L_{\text{YM}} - ig \int (dx) \bar{\psi}^A_i(x) (T^a)^{AB} (\gamma_\mu)_{ij} \cdot \psi^B_j(x) \frac{\delta}{\delta A^a_\mu(x)} \right] F[A] \right\rangle. \tag{7.1}$$

This equation is true at all λ. Our strategy is to show that the fermionic current term $\bar{\psi} T^a \gamma_\mu \psi$ of (7.1) may be replaced, at large λ, by the determinantal term of (2.2).

The next step is the observation that the SD λ-family (5.2) has an explicit factor of λ multiplying all fermionic terms (except the fermionic-Zwanziger gauge-fixing). The no-growth theorem tells us that these terms must therefore be set to zero

$$\int (dx)(dy)\Big\langle\Big\{[(\not{D}^2-m^2)^{AB}_{ij}]_{xy}\,\psi^B_j(y)\,\frac{\delta}{\delta\psi^A_i(x)} + [\{(\not{D}^2-m^2)^T\}^{AB}_{ij}]_{xy}\,\bar{\psi}^B_j(y)\,\frac{\delta}{\delta\bar{\psi}^A_i(x)}\Big\}\,G[A,\psi,\bar{\psi}]\Big\rangle_{(0)}$$
$$=2\int (dx)(dy)\Big\langle[(\{\not{D}-m\}\,\mathbb{R}^2)^{AB}_{ij}]_{xy}\cdot\frac{\delta}{\delta\bar{\psi}^B_j(y)}\,\frac{\delta}{\delta\psi^A_i(x)}\,G[A,\psi,\bar{\psi}]\Big\rangle_{(0)} \quad (7.2)$$

in the large λ limit, in order to prevent the growth of the other terms. In (7.2), the superscript T denotes transpose while the subscript (0) on an equation is a reminder that the equation is only true in the large λ limit.

The λ-independent equations (7.1) and (7.2) together are accurately termed *the SD equations at large* λ. Notice that the fermionic-Zwanziger terms do not appear in the SD equations at large λ, as anticipated in Sects. 4, 5, and 6. We also remark that these equations are clearly true at the unregularized and ungauge-fixed action level, and so form – in their own right – an adequately gauge-fixed and regularized version of the original theory.

It is a remarkable fact that (7.2) is a disguised version of the integrated fermionic prescription (2.5). To see this, it is necessary to expand $G[A,\psi,\bar{\psi}]$ in fermionic moments. As a first example, consider the case of one fermion pair,

$$G_1[A,\psi,\bar{\psi}] = \int (du)(dv)[\Omega_1^{-1}]^{Aix;Bjy}_{Clu;Dmv}\,\psi^C_l(u)\,\bar{\psi}^D_m(v)\,F_1[A], \quad (7.3)$$

in which Ω_1^{-1} is defined by

$$\int (du)(dv)[\Omega_1^{-1}]^{Aix;Bjy}_{Clu;Dmv}[\Omega_1]^{Clu;Dmv}_{Ekz;Fnw} = \delta^A_E\,\delta^i_k\,\delta(x-z)\,\delta^B_F\,\delta^j_n\,\delta(y-w), \quad (7.4\text{a})$$

$$[\Omega_1]^{Aix;Bjy}_{Clu;Dmv} \equiv [(\not{D}^2-m^2)^{AC}_{il}]_{xu}\,\delta^{BD}\,\delta_{jm}\,\delta(y-v) + [\{(\not{D}^2-m^2)^T\}^{BD}_{jm}]_{yv}\,\delta^{AC}\,\delta_{il}\,\delta(x-u). \quad (7.4\text{b})$$

The operator Ω_1^{-1} is well defined since Ω_1 has no zero eigenvalues when $m \neq 0$. Substituting G_1 into (7.2), we compute

$$\langle\psi^A_i(x)\,\bar{\psi}^B_j(y)\,F_1[A]\rangle_{(0)} = 2\int (du)(dv)\langle[\Omega_1^{-1}]^{Aix;Bjy}_{Clu;Dmv}[(\{\not{D}-m\}\,\mathbb{R}^2)^{CD}_{lm}]_{uv}\rangle_{(0)} = \langle[(\{\not{D}+m\}^{-1}\,\mathbb{R}^2)^{AB}_{ij}]_{xy}\,F_1[A]\rangle_{(0)}, \quad (7.5)$$

where the identity

$$[(\{\not{D}-m\}\,\mathbb{R}^2)^{CD}_{lm}]_{uv} = \tfrac{1}{2}\int (dy)(dz)[\Omega_1]^{Clu;Dmv}_{Eky;Fnz}[(\{\not{D}+m\}^{-1}\,\mathbb{R}^2)^{EF}_{kn}]_{yz} \quad (7.6)$$

has been used to obtain the last step in (7.5). The large λ result (7.5) is precisely the one-pair fermionic prescription (2.6a) of the integrated regularization.

As a special case of (7.5), we have

$$-ig\Big\langle\bar{\psi}(x)\,T^a\gamma_\mu\,\psi(x)\,\frac{\delta F[A]}{\delta A^a_\mu(x)}\Big\rangle_{(0)} = ig\Big\langle\mathrm{Tr}[T^a\gamma_\mu(\{\not{D}+m\}^{-1}\,\mathbb{R}^2)_{xx}]\,\frac{\delta F[A]}{\delta A^a_\mu(x)}\Big\rangle_{(0)}, \quad (7.7)$$

which states that, at large λ, the fermion current term in (7.1) is the determinantal term in (2.2) of the integrated regularization. This completes the derivation of the integrated SD equations (2.2) as the large λ limit of the regularized SIAG λ-family (5.2).

In general, with G_n an n-fermion pair moment, (7.2) relates an n-pair Green function to a Green function with $(n-1)$ pairs (from the fermionic functional Laplacian). The complete moment expansion in the sector with zero fermion number may therefore be obtained by induction based on the result (7.5). The explicit moments G_n are

$$G_n[A,\psi,\bar{\psi}] = \int \prod_{r=1}^{n}(dx_r)(dy_r)[\Omega_n^{-1}]^{A_1i_1u_1,\dots,A_ni_nu_n;B_1j_1v_1,\dots,B_nj_nv_n}_{C_1l_1x_1,\dots,C_nl_nx_n;D_1m_1y_1,\dots,D_nm_ny_n}\cdot\psi^{C_1}_{l_1}(x_1)\dots\psi^{C_n}_{l_n}(x_n)\,\bar{\psi}^{D_1}_{m_1}(y_1)\dots\bar{\psi}^{D_n}_{m_n}(y_n)\,F_n[A], \quad (7.8)$$

in which the operator Ω_n is defined by

$$[\Omega_n]^{A_1i_1u_1,\dots,A_ni_nu_n;B_1j_1v_1,\dots,B_nj_nv_n}_{C_1l_1x_1,\dots,C_nl_nx_n;D_1m_1y_1,\dots,D_nm_ny_n} \equiv \sum_{r=1}^{n}[D_n]^{A_ri_ru_r;B_rj_rv_r}_{C_rl_rx_r;D_rm_ry_r}, \quad (7.9\text{a})$$

$$[D_n]^{A_ri_ru_r;B_rj_rv_r}_{C_rl_rx_r;D_rm_ry_r} \equiv [\Omega_1]^{A_ri_ru_r;B_rj_rv_r}_{C_rl_rx_r;D_rm_ry_r}\cdot\prod_{p\neq r}^{n}\delta^{A_p}_{C_p}\,\delta^{i_p}_{l_p}\,\delta(u_p-x_p)\,\delta^{B_p}_{D_p}\,\delta^{j_p}_{m_p}\,\delta(v_p-y_p). \quad (7.9\text{b})$$

Although we omit further details for brevity, the result of this straightforward inductive procedure is precisely the integrated fermionic prescription (2.5) in the sector with zero fermion number.

Finally, the content of (7.2) is exhausted in a similar inductive process which implies the vanishing of any Green function with non-zero fermion number. In this way, one completes the proof that the large λ limit of the SIAG λ-family (5.2) is exactly the integrated regularized SD system (2.2, 5).

We have also studied the integration of the SD system (5.2) at finite λ, but the situation in this case seems prohibitively complex.

8. Remarks on the Regularized Integration Technique

As another example of the regularized integration technique, we consider the λ-family of regularized SD equations for scalar electrodynamics

$$0=\int(dx)\left\langle\left\{\left[-\frac{\delta S^{(0)}}{\delta A_\mu(x)}+\int(dy)\,R^2_{xy}(\Box)\frac{\delta}{\delta A_\mu(y)}\right.\right.\right.$$
$$\left.+ie[\phi(x)D^*_\mu\phi^*(x)-\phi^*(x)D_\mu\phi(x)]\right]\frac{\delta}{\delta A_\mu(x)}$$
$$+\lambda\left[-\frac{\delta S}{\delta\phi(x)}+\int(dy)\,R^2_{xy}(\Delta^*)\frac{\delta}{\delta\phi(y)}\right]\frac{\delta}{\delta\phi^*(x)}$$
$$+\lambda\left[-\frac{\delta S}{\delta\phi^*(x)}+\int(dy)\,R^2_{xy}(\Delta)\frac{\delta}{\delta\phi^*(y)}\right]\frac{\delta}{\delta\phi(x)}$$
$$\left.\left.-Z(x)\,G(x)\right\}F[A,\phi,\phi^*]\right\rangle, \tag{8.1}$$

the $\lambda=1$ case of which was given in [3]. The implied SD equations at large λ are

$$0=\int(dx)\left\langle\left[-\frac{\delta S^{(0)}}{\delta A_\mu(x)}+\partial_\mu Z(x)\right.\right.$$
$$+\int(dy)\,R^2_{xy}(\Box)\frac{\delta}{\delta A_\mu(y)}+ie[\phi(x)D^*_\mu\phi^*(x)$$
$$\left.\left.-\phi^*(x)D_\mu\phi(x)]\right]\frac{\delta}{\delta A_\mu(x)}F[A]\right\rangle, \tag{8.2a}$$

$$0=\int(dx)\left\langle\left[-\frac{\delta S}{\delta\phi(x)}\frac{\delta}{\delta\phi^*(x)}-\frac{\delta S}{\delta\phi^*(x)}\frac{\delta}{\delta\phi(x)}\right.\right.$$
$$\left.\left.+2\int(dy)\,R^2_{xy}(\Delta)\frac{\delta}{\delta\phi^*(y)}\frac{\delta}{\delta\phi(x)}\right]G[A,\phi,\phi^*]\right\rangle_{(0)}, \tag{8.2b}$$

where (8.2b) follows from a no-growth theorem. As in Sect. 7, a matter-field moment expansion* of (8.2b) yields the integrated prescription (2.11), the one-pair form of which is

$$\langle\phi(x)\phi^*(y)F_1[A]\rangle_{(0)}=\langle[\Delta^{-1}R^2]_{xy}F_1[A]\rangle_{(0)}. \tag{8.3}$$

Then, (8.2a) at large λ is recognized as the SD (2.9) of the integrated regularized formulation.

* For example, the one-pair moment operator is $[\Omega_1]^{x;y}_{u;v}=\Delta_{xu}\delta(y-v)+\Delta_{vy}\delta(x-u)$

We also mention the "naive" formulation [6] of gauge theory fermions, in which a dimensionful parameter λ naturally appears. Assuming a no-growth theorem, this system is also equivalent at large λ to the integrated regularized formulation (2.2, 5).

Finally, an integrated formulation for chiral fermions may be devised with a determinantal term

$$ig\int(dx)\,\mathrm{Tr}\left[T^a\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\{\not{D}^{-1}\mathbb{R}^2(\not{D}^2)\}_{xx}\right]\delta/\delta A^a_\mu(x)$$

in the SD operator of (2.2). With attention to zero modes, such a formulation should be adequate for the study of perturbative chiral anomalies. In the case of theories with global anomalies [13] however, an integrated formulation is problematic (reflecting problems in the definition of chiral determinants), and should be approached through a non-perturbative study of the large-λ regularized SIAG systems.

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